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On the application of the Kelvin–Arnol'd energy principle to the stability of forced two-dimensional inviscid flows

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Arnol'd developed two distinct yet closely related approaches to the linear stability of Euler flows. One is widely used for two-dimensional flows and involves constructing a conserved functional whose first variation vanishes and whose second variation determines the linear (and nonlinear) stability of the motion. The second method is a refinement of Kelvin's energy principle which states that stable steady Euler flows represent extremums in energy under a virtual displacement of the vorticity field. The conserved-functional (or energy-Casimir) method has been extended by several authors to more complex flows, such as planar MHD flow. In this paper we generalize the Kelvin–Arnol'd energy method to two-dimensional inviscid flows subject to a body force of the form $-\Phi\nabla f$. Here Φ is a materially conserved quantity and f an arbitrary function of position and of Φ. This encompasses a broad class of conservative flows, such as natural-convection planar and poloidal MHD flow with the magnetic field trapped in the plane of the motion, flows driven by electrostatic forces, swirling recirculating flow, self-gravitating flows and poloidal MHD flow subject to an azimuthal magnetic field. We show that stable steady motions represent extremums in energy under a virtual displacement of Φ and of the vorticity field. That is, $\mathrm{d}^1E = 0$ at equilibrium and whenever d^2E is positive or negative definite the flow is (linearly) stable. We also show that unstable normal modes must have a spatial structure which satisfies $\mathrm{d}^2E = 0$. This provides a single stability test for a broad class of flows, and we describe a simple universal procedure for implementing this test. In passing, a new test for linear stability is developed. That is, we demonstrate that stability is ensured (for flows of the type considered here) whenever the Lagrangian of the flow is a maximum under a virtual displacement of the particle trajectories, the displacement being of the type normally associated with Hamilton's principle. A simple universal procedure for applying this test is also given. We apply our general stability criteria to a range of flows and recover some familiar results. We also extend these ideas to flows which are subject to more than one type of body force. For example, a new stability criterion is obtained (without the use of Casimirs) for natural convection in the presence of a magnetic field. Nonlinear stability is also considered. Specifically, we develop a nonlinear stability criterion for planar MHD flows which are subject to isomagnetic perturbations. This differs from previous criteria in that we are able to extend the linear criterion into the nonlinear regime. We also show how to extend the Kelvin–Arnol'd method to finite-amplitude perturbations.

1. Introduction

1.1. *Two-dimensional forced flows*

There are many branches of fluid mechanics, such as natural convection, magneto-hydrodynamics and electrodynamics, in which a fluid is subjected to a body force of the form $-\Phi\nabla f$, where Φ is a materially conserved quantity. The function f is often, though not always, the potential of some imposed field. Less often we encounter forces of the form $f\nabla\Phi$ where Φ is, again, materially conserved. In either case, we may write the inviscid equation of motion in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\Omega} - \nabla C + f\nabla\Phi; \quad f = f(\boldsymbol{x}, \Phi). \tag{1.1}$$

Here $\boldsymbol{\Omega}$ is the vorticity, $\boldsymbol{u}$ is the velocity, f is an arbitrary function of position and of Φ (but not an explicit function of time), and C is a generalized Bernoulli function. We shall restrict ourselves to cases where the fluid is incompressible and Φ vanishes on the boundary. We then have the auxiliary equations

$$\nabla \cdot \boldsymbol{u} = 0; \qquad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad S \tag{1.2}$$

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = 0; \qquad \Phi = 0 \quad \text{on} \quad S. \tag{1.3}$$

In this paper we focus on confined two-dimensional flows. In particular, we are interested in the existence of steady solutions of (1.1), their structure, and above all their stability characteristics. These flows may be planar, lying in the (x, y)-plane, or else axisymmetric, lying in the (r, z)-plane. (In the latter case we adopt cylindrical polar coordinates (r, θ, z).) In any event, all streamlines are assumed to be closed with the flow confined to a simply connected domain, V, and bounded by the surface S.

We shall place two restrictions on our analysis. First, we limit the discussion to linear two-dimensional stability. That is, any disturbance is assumed to have an infinitesimal amplitude and to be strictly two-dimensional (or poloidal) in form. Second, we consider only non-dissipative systems in which energy is conserved. Specifically, we ignore all viscous effects, such as cross-stream diffusion, boundary layers, Ekman pumping, or Hartmann layers. This is, perhaps, the more severe restriction. (We lift the restriction of small-amplitude perturbations only in the penultimate section, where we touch upon the question of nonlinear stability.)

The range of flows governed by (1.1)–(1.3) is surprisingly broad. Some familiar examples are:

(i) (planar) natural convection of an incompressible fluid which exhibits a spatial variation of density, ρ, perhaps due to 'frozen-in' variations of temperature or chemistry;

(ii) MHD flow of a perfectly conducting fluid in which both the trapped magnetic field, $\boldsymbol{B}$, and the velocity lie in the (x, y)-plane;

(iii) a non-conducting medium which contains bound charges (say charged dust particles in air) and which moves in the (x, y)-plane under the influence of mutual electrostatic forces;

(iv) axisymmetric swirling recirculating flow in which the poloidal component of motion $(u_r, 0, u_z)$ is subject to the centripetal acceleration associated with the angular momentum $\Gamma = ru_\theta$;

(v) a self-gravitating poloidal flow in the Boussinesq approximation (i.e. where the departure in density from the mean is small, $\rho' = \rho - \bar{\rho} \ll \bar{\rho}$);

(vi) axisymmetric MHD flow in which $\boldsymbol{u}$ is poloidal $(u_r, 0, u_z)$ and $\boldsymbol{B}$ is azimuthal $(0, B_\theta, 0)$;

(vii) axisymmetric MHD flow in which both $\boldsymbol{u}$ and $\boldsymbol{B}$ are poloidal.

In the interests of brevity we shall restrict ourselves to these seven examples. However, it should be emphasized that this list is by no means exhaustive. We could, for example, extend case (iii) to ionized plasmas, or case (vi) to compressible MHD flow with $\boldsymbol{B}$ normal to the plane of motion. (In either case we must, of course, modify the auxiliary equations to allow for compressibility.) Moreover, as shown in §8, our ideas extend to flows which are subject to a combination of forces. Thus, for example, we examine the stabilization of a Rayleigh–Taylor instability by a magnetic field. However, examples (i)–(vii) seem fairly representative, and so for the most part we confine attention to these. Note that all of the examples listed above represent conservative systems. The exact form of f and Φ for each of these flows, as well as a corresponding justification of (1.1)–(1.3), is given in §3.

We shall examine the stability characteristics of these forced flows using certain variational techniques. Indeed, one of our observations is that the Kelvin–Arnol'd energy principle, which states that stable steady flows represent extremums in energy, may be extended to encompass steady solutions of (1.1). (This method is commonly used to examine the stability of (unforced) Euler flows (Arnol'd 1966*b*).) We do not claim that *all* of the stability results obtained in this way are new. On the contrary, many of these flows have been studied using other (energy-like) methods, most notably by Holm *et al.* (1985), Shepherd (1992) and Vladimirov, Moffatt & Illin (1996). We do claim, however, that our systematic extension of the Kelvin–Arnol'd energy principle to forced flows is both new and useful. The utility of this approach lies in the fact that it furnishes a single stability test for a wide range of flows, and that there is a simple universal procedure for implementing this test.

1.2. *Arnol'd's stability theorems*

Now Arnol'd developed two related but distinct approaches to the linear stability of two-dimensional unforced Euler flows. It is important that, from the outset, we distinguish between the two, and so we briefly describe both methods here. A more detailed discussion is given in §2. The first method (Arnol'd 1966*a*) is commonly referred to as the *conserved functional, energy-Casimir,* or *formal stability* approach. It is widely used for two-dimensional flows and its attraction lies in the fact that it is readily generalized to give nonlinear results. (This method has been extended to other categories of two-dimensional flow by Holm *et al.* 1985, Shepherd 1992 and many others.) The second approach is commonly referred to as *Arnold's variational principle*, or *the Kelvin–Arnol'd energy principle*, and its power lies in its generality. It is not restricted to two-dimensional flows, but may be applied to three-dimensional Euler flows (Arnol'd 1966*a*). Now these two methods define stability in rather different ways (involving different classes of perturbations), and so one question which may be asked at the outset is: do they share a common view of stability and so lead to compatible stability criteria? We shall see that, as suggested by Holm *et al.* (1985) and Shepherd (1992), the answer to this question is yes and we return to this issue in §2.

The starting point for establishing *formal stability* is to construct a conserved functional of the form $A = E - I\,(\Omega)$, where E is the global kinetic energy and $I\,(\Omega)$ is an integral invariant of the vorticity field (called a Casimir). One then chooses $I\,(\Omega)$ such that A is stationary at equilibrium. That is, if Ω_o is the vorticity of some steady flow, and ω is an arbitrary infinitesimal perturbation of Ω_o $(\Omega = \Omega_o + \omega)$, then the first-order change in A is zero, $\delta^1 A = 0$. The second variation of A then provides information about the stability of the flow. Specifically, the flow is said to possess *formal stability* if $\delta^2 A$ is positive or negative definite (Holm *et al.* 1985). We shall return to this topic in §2, where we explain the significance of $\delta^2 A$ being single signed. In the meantime, we consider Arnold's second approach to stability.

Arnold's variational (or energy) principle is distinct from, but related to, his conserved functional approach. It is not restricted to two-dimensional flows (i.e. flows with only one vorticity component). Indeed it has been applied to axisymmetric swirling recirculating flows by Davidson (1994) where the introduction of a third velocity component has a profound effect on stability. The variational (energy) method rests on the idea of applying a Lagrangian displacement to the fluid particles and examining the consequent change in kinetic energy. Specifically, Arnol'd's variational principle says the following. Suppose the vorticity distribution of some steady Euler flow is perturbed by a volume-preserving virtual displacement field, $\boldsymbol{\eta}(\boldsymbol{x})$. During this perturbation the vorticity is materially conserved (in a sense to be made precise later) and consequently such a perturbation is referred to as *isovortical*. Now consider the change in global kinetic energy which arises from this perturbation. Let $\mathrm{d}^1 E$ and $\mathrm{d}^2 E$ be the first- and second-order changes in E. (We use d rather than δ here to distinguish clearly between the two kinds of perturbation which, as we shall see, are quite different in form.) Then the Kelvin–Arnol'd variational principle states that:

(i) E is stationary, i.e. $\mathrm{d}^1 E = 0$ for all $\boldsymbol{\eta}$;

(ii) when $\mathrm{d}^2 E$ is positive or negative definite, the flow is linearly stable;

(iii) when $\mathrm{d}^2 E$ is not of definite sign then the flow could be (but need not be) unstable, and any exponentially growing normal mode must have a spatial structure which ensures that $\mathrm{d}^2 E = 0$ at all times.

Conditions (i) and (ii) were stated without proof by Kelvin (1887). However, the first rational argument for these propositions was given by Arnol'd (1966*b*). Note that, in this approach, all integrals of the type $I(\Omega)$ are automatically conserved through the particular choice of the d-perturbation. One advantage of this method is that it obviates the need to find an integral $I(\Omega)$ which ensures that the first variation in A vanishes. As noted by Holm *et al.* (1985), and as demonstrated explicitly in §2, $I(\Omega)$ is a form of Lagrange multiplier which allows one to relax the class of admissible variations from d-perturbations to δ-perturbations.

In summary, then, Arnol'd's variational principle states that an Euler flow is (linearly) stable whenever E is a maximum or a minimum with respect to all possible isovortical perturbations. To broaden Arnol'd's energy principle from unforced to forced flows we merely need to generalize the concept of an isovortical perturbation.

1.3. *Extension of the Kelvin–Arnol'd energy theorem to forced flows*

Our starting point is to note that, while Ω is materially conserved in a planar unforced flow, this is no longer the case for forced flows of the type (1.1). Rather, Φ is materially conserved. However, Ω is globally constant in the sense that the total vorticity contained within each contour of constant Φ (a Φ-line) is conserved by (1.1). (See §4.) This suggests that we try a new type of perturbation which mimics the dynamics of these forced flows. As before, we consider a volume-preserving virtual displacement field, $\boldsymbol{\eta}$. This time, however, we apply it to Φ, rather than to Ω. Simultaneously, we rearrange the vorticity field in such a way that the integral of Ω within each closed Φ-line is preserved during the perturbation.

In §2 and §4, we introduce the concept of function space, and in particular the idea (initiated by Arnol'd 1996*b*) that an inviscid flow evolves within a certain subdomain of the space of all solenoidal velocity fields. In the case of Euler flows these subdomains are characterized by the fact that the vorticity fields can be mapped one to another by a smooth volume-preserving displacement. For forced flows these subdomains are more complex. They are composed of $\boldsymbol{u}$ and Φ fields in which various distributions of Φ may be mapped one to another by a smooth displacement while simultaneously conserving the integral of Ω within each Φ-line. We shall use this to argue that our new type of perturbation plays a role in the stability of forced flows which is precisely equivalent to the role played by Arnol'd's isovortical perturbation in the stability of unforced Euler flows. Indeed, it is a convenient shorthand to introduce the term *generalized isovortical perturbation*, or simply *isovortical perturbation*, to denote any perturbation of Φ and Ω which satisfies the conditions above, that is, any perturbation in which: (i) Φ is materially conserved during the perturbation; (ii) the total vorticity contained within each Φ contour is conserved.

Now consider the energy of a forced flow governed by (1.1). The total energy will consist of kinetic energy plus the energy associated with the body force, which might be magnetic energy, electrostatic energy, or gravitational energy. We now make the following proposition concerning the variation of the total energy, E.

PROPOSITION 1. *If a solution of (1.1)–(1.3) is perturbed by a generalized isovortical perturbation (as defined above), then:*

(*i*) *E is stationary* $\left(\mathrm{d}^1 E = 0\right)$ *whenever the flow is a steady solution of (1.1);*

(*ii*) *whenever the energy is a maximum or a minimum with respect to all possible isovortical perturbations, the flow is linearly stable;*

(*iii*) *if* $\mathrm{d}^2 E$ *is of indefinite sign then the flow could be (but need not be) unstable, and any exponentially growing normal mode must have a spatial structure which ensures that* $\mathrm{d}^2 E = 0$ *at all times.*

In short, we claim that, by adjusting the definition of an isovortical perturbation, the Kelvin–Arnol'd energy principle may be extended to encompass forced inviscid flows governed by (1.1). We shall prove this only for the seven categories of flow listed above. Nevertheless, we believe, but do not prove, that Proposition 1 is valid for all conservative, steady solutions of (1.1). This belief rests on the idea that unsteady solutions of (1.1) trace out constant energy 'contours' within the appropriate subdomain of function space. If, in the vicinity of a steady solution, these contours are restricted to a nested set of closed shells then, in some sense, a perturbed flow cannot migrate far from the equilibrium solution. It is natural, therefore, to expect stable flows to represent a maximum or minimum in energy within such a subspace. Saddle points, on the other hand, would normally represent unstable flows. In Hamiltonian mechanics these subspaces are referred to as *iso-Casimir* surfaces. We may regard Proposition 1 as defining and exploring these surfaces. We shall return to this viewpoint in §5.

1.4. *Structure of this paper*

We conclude this introduction by outlining the structure of the paper. In §2 we return to the topic of Arnol'd's two stability theorems. This allows us to formally introduce a number of fundamental concepts, such as isovortical perturbations and virtual displacement fields. It also sets the scene for the subsequent analysis of forced flows. Next, in §3, we review the broad range of forced inviscid flows which are governed by (1.1). In each case we identify the form of f and Φ. In §4 we outline the general properties of (1.1). In particular, we show that (1.1) supports non-trivial steady flows, conserves energy provided f is not an explicit function of time and supports integral invariants of the form

$$I(\Omega, \Phi) = \int [\Omega g(\Phi) + h(\Phi)] \, \mathrm{d}V. \tag{1.4}$$

Here g and h are smooth but otherwise arbitrary functions of Φ. Invariants of the form of (1.4) are central to the energy-Casimir (conserved functional) approach to stability.

In §5, we adopt the variational (or energy) route to stability, whereby steady solutions of (1.1) are subject to a generalized isovortical perturbation. Subsequently, in §6, we apply the energy-Casimir (conserved functional) method to (1.1). On comparing the results of §5 and §6 we find that, for each type of flow, the stability characteristics obtained via the energy-Casimir method are identical to those predicted by Proposition 1. We take this as proof of Proposition 1. That is, Arnol'd's energy principle may be extended (in the manner suggested) to forced inviscid flows of type (i)–(vii). (In the language of Hamiltonian mechanics, this indicates that our definition of a generalized isovortical perturbation is just sufficient to identify the iso-Casimir surfaces (or *isovortical leaves*) in function space.)

It is natural to enquire whether or not there is a simple relationship between the Kelvin–Arnol'd energy principle and Hamilton's principle of least action. In §7 we show that there is. Specifically, stable steady flows have a *Lagrangian* which is a maximum under a virtual displacement of the particle trajectories. This is, in effect, a principle of maximum (rather than least) action.

Next, in §8 and §9, we provide a detailed physical interpretation of the stability characteristics obtained for flows (i)–(vii). To some extent, this is performed on a case-by-case basis. However, there is one identifiable trend. Whenever f is a prescribed function of position, d^2E is indefinite in sign. This is true for natural convection (case i) MHD flow where $\boldsymbol{B}$ is azimuthal (case vi), and swirling recirculating flows (case iv). We argue that the ambiguity in the sign of d^2E is indicative of a true instability of a type related to the Rayleigh–Taylor instability. We conclude §8 by showing how to extend our ideas to flows driven by a combination of forces. For example, we look at natural convection in the presence of a magnetic field. Here we obtain a new stability criterion.

In §10, we discuss nonlinear stability. Here we use a finite-amplitude conservation theorem to derive a new nonlinear stability criterion for planar MHD flows. We also show how to extend the Kelvin–Arnol'd method to nonlinear problems.

2. A review of Arnol'd's stability theorems

We now review Arnol'd's stability theorems for (unforced) Euler flows. We consider both his variational (energy) and conserved functional (energy-Casimir) approaches to stability. Our purpose is to establish the relationship between the two methods, and to introduce the fundamental tools of the variational method, particularly the concept of virtual displacement fields. The review is brief. However, more details may be found in Arnol'd (1966*a, b*), Moffatt (1986) and Holm *et al.* (1985). We might note in passing that much of the current literature on the energy-Casimir method is written in terms of Hamiltonian mechanics. That is, the governing equation ((1.1) in our case) is written in symplectic Hamiltonian form and non-canonical Poisson brackets introduced which determine the integral invariants, i.e. Casimirs. However, the Hamiltonian and integral invariants of (1.1) are readily identified by inspection.

Consequently, there is no particular need to adopt the Hamiltonian formulation and we do not do so in this paper.

2.1. *Arnol'd's conserved-functional theorem and variational principle*

For the sake of brevity, we shall consider only planar motion. In the steady state, the Euler equation reduces to the statement $\Omega = -C'(\Psi)$, where C is Bernoulli's function $\left(p/\rho + \frac{1}{2}u^2\right)$, and the streamfunction, Ψ, is defined via $\boldsymbol{u} = \nabla \times (\Psi \hat{\boldsymbol{e}}_z)$. Steady flows are therefore governed by

$$\nabla^2 \Psi = -\Omega(\Psi) = C'(\Psi). \tag{2.1}$$

We now review each of Arnol'd's stability theorems as applied to (2.1). We turn first to the variational (or energy) principle. Here our starting point is the concept of function space, and in particular the space of all solenoidal velocity fields which satisfy $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on S. Adjacent points in function space represent similar velocity fields, and so an unsteady Euler flow may be considered to trace out a curve (evolve) in this infinite-dimensional space. This evolution is characterized by the material conservation of vorticity and by the conservation of global kinetic energy. It is convenient, therefore, to divide the function space into subdomains in which the vorticity fields may be mapped one to another by a smooth volume-preserving displacement of the vorticity. Such subdomains are referred to as *isovortical sheets* or *leaves.*

An Euler flow is constrained to follow a constant energy trajectory or contour on an isovortical sheet. This is illustrated schematically in figure 1(a), where function space is represented as three-dimensional. It follows that steady Euler flows represent stationary points on these sheets; that is, points at which $\mathrm{d}^1E = 0$. If this point is also an extremum in energy, so that constant-energy curves are locally elliptic, then in some sense the flow is stable. That is, an isovortically perturbed flow will subsequently evolve on a constant-energy contour which always lies 'close' to its starting point. To clarify this idea, we use the following argument. The energy E is conserved by any flow. Moreover, on an isovortical sheet the first variation, d^1E, vanishes at equilibrium. It follows that, in the linear approximation, d^2E is conserved by a disturbance. Now suppose that $\|\mathrm{d}\boldsymbol{u}\|$ is the distance (in function space) between the perturbed flow, $\boldsymbol{u} = \boldsymbol{u}_o + \mathrm{d}\boldsymbol{u}$, and the steady flow whose stability is in question: $\|\mathrm{d}\boldsymbol{u}\|^2 = \int (\boldsymbol{u} - \boldsymbol{u}_o)^2\,\mathrm{d}V$. We now take $\|\mathrm{d}\boldsymbol{u}\|^2$ as a measure of the disturbance. Then, if the flow is to be unstable (in our chosen norm), $\|\mathrm{d}\boldsymbol{u}\|^2$ will grow with time despite the conservation of d^2E. In such cases the ratio $\mathrm{d}^2E/\,\|\mathrm{d}\boldsymbol{u}\|^2$ must tend to zero. (Note that both d^2E and $\|\,\mathrm{d}\boldsymbol{u}\,\|^2$ are quadratic in the disturbance.) Consequently, provided this ratio can be bound away from zero, the flow cannot become unstable. The implication is that, whenever d^2E is positive or negative definite, the flow must be stable. Indeed, we could *define* stability as an extremum in energy under an isovortical perturbation. Of course, one would wish to demonstrate that this definition coincides with more conventional notions of linear stability. A slightly more cautious approach is taken in our generalized treatment of (1.1). That is, we *assert* that stability is associated with extremums in E under a *generalized isovortical perturbation* and then demonstrate that this is compatible with more familiar notions of stability.

Now suppose that the equilibrium flow represents not an extremum but rather a saddle point. Then adjacent energy contours diverge, so that an isovortically perturbed flow is no longer constrained to stay close to the initial equilibrium position. Such a flow could be, and probably is, unstable. If perturbed it will move off along (or remain close to) one of the separatrices, which is characterized by $\mathrm{d}^2E = 0$. In summary, then,

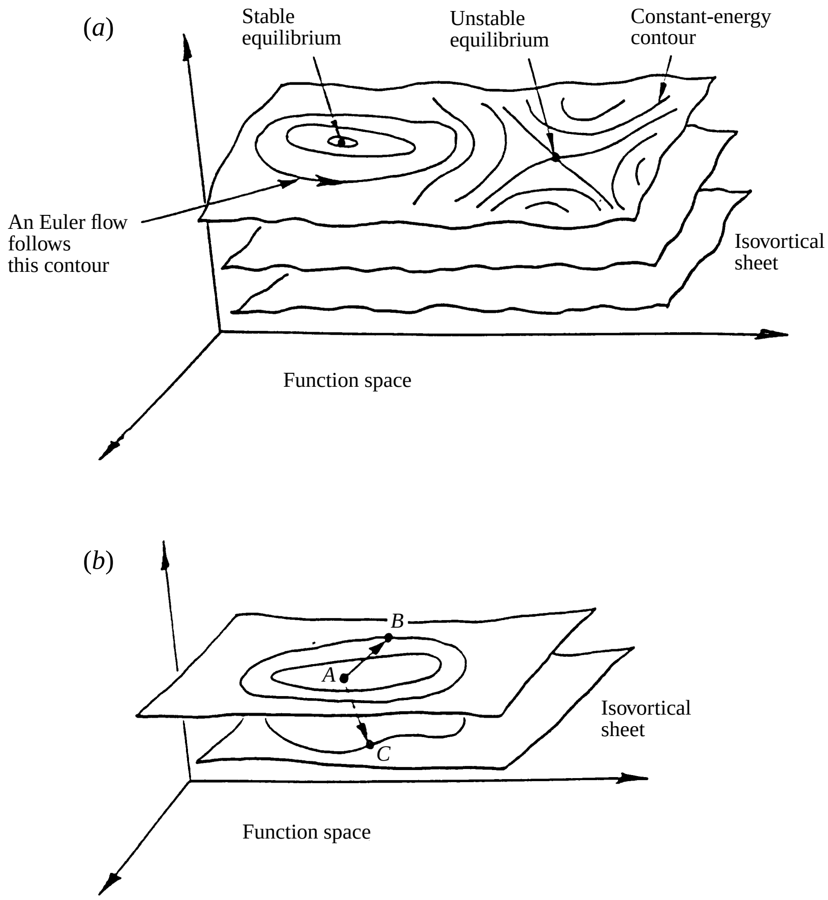

FIGURE 1. (*a*) Unsteady Euler flows follow constant-energy contours on an isovortical sheet. Stable equilibria are represented by extremums in energy. (*b*) Virtual displacement AB is isovortical, while perturbation AC is not. The Kelvin–Arnol'd energy method considers perturbations of type AB, while the conserved-functional approach considers perturbations of type AC.

we may associate stability with extremums in E under an isovortical perturbation. Unstable modes, on the other hand, are associated with $\mathrm{d}^2E = 0$. This is, in effect, Arnol'd's energy principle. It represents a sufficient, though not necessary, condition for stability.

Now suppose we wish to determine the stability of some particular flow using Arnol'd's variational principle. We need to devise some means of evaluating d^2E under an isovortical perturbation. This leads to the concept of a virtual displacement field (Moffatt 1986). As in Lagrangian dynamics, such a displacement simply represents a perturbation of the generalized coordinates, and is quite arbitrary except to the extent that the system constraints must be satisfied. In the case of Euler flows, these constraints are conservation of volume and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on S. In order to ensure that volume is conserved during a virtual displacement, it is convenient to suppose that the displacement occurs through the action of some solenoidal velocity field, $\boldsymbol{v}(\boldsymbol{x})$, applied for a short time τ (Moffatt 1986). If we define the displacement field $\boldsymbol{\eta}(\boldsymbol{x})$ through $\boldsymbol{\eta} = \boldsymbol{v}\tau$, then we have

$$\nabla\cdot\boldsymbol{\eta} = 0, \quad \boldsymbol{\eta}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad S.$$

An isovortical perturbation is achieved by applying the virtual displacement field, $\boldsymbol{\eta}$, to the vorticity of a steady Euler flow. That is, we consider Ω to be advected by $\boldsymbol{v}$ for a short time, τ. This allows us to migrate across the isovortical sheet and indeed we may think of $\boldsymbol{\eta}$ as providing a local coordinate in function space for each flow adjacent to an equilibrium. The first- and second-order changes in Ω and $\boldsymbol{u}$ are readily shown to be (see Davidson 1994, or Moffatt 1986)

$$\mathrm{d}^1\Omega = -\boldsymbol{\eta}\cdot\nabla\Omega; \quad \mathrm{d}^1\boldsymbol{u} = \boldsymbol{\eta}\times\boldsymbol{\Omega} + \nabla\phi_1, \tag{2.2}$$

$$\mathrm{d}^2\Omega = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla\left(\mathrm{d}^1\Omega\right); \quad \mathrm{d}^2\boldsymbol{u} = \tfrac{1}{2}\left(\boldsymbol{\eta}\times\mathrm{d}^1\boldsymbol{\Omega}\right) + \nabla\phi_2, \tag{2.3}$$

where ϕ_1 and ϕ_2 are chosen to ensure that $\mathrm{d}^1\boldsymbol{u}$ and $\mathrm{d}^2\boldsymbol{u}$ are solenoidal. Now we are interested in how E varies as we probe the surrounding function space. It is readily confirmed that, in accordance with Arnol'd's theorem, d^1E vanishes if $\boldsymbol{u}$ represents a steady Euler flow. The second-order change in E is

$$\mathrm{d}^2E = \frac{1}{2}\int_V \left[\left(\mathrm{d}^1\boldsymbol{u}\right)^2 + 2\boldsymbol{u}\cdot\mathrm{d}^2\boldsymbol{u}\right]\mathrm{d}V = \frac{1}{2}\int_V \left[\left(\mathrm{d}^1\boldsymbol{u}\right)^2 + \boldsymbol{\eta}\cdot\left(\mathrm{d}^1\boldsymbol{\Omega}\times\boldsymbol{u}\right)\right]\mathrm{d}V. \tag{2.4}$$

Now suppose that Ω_o and Ψ_o are the vorticity and streamfunction of the steady Euler flow under investigation. Then from (2.2), our expression for d^2E simplifies to

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\psi)^2 - \left(\nabla^2\psi\right)^2/\Omega_o'(\Psi_o)\right]\mathrm{d}V \tag{2.5}$$

where ψ is the first-order change in Ψ_o ($\Psi = \Psi_o + \psi$).

Expression (2.5) was first obtained by Arnol'd (1966*b*), and it furnishes some useful linear stability criteria. The variational principle asserts that a flow is stable in the energy norm whenever d^2E is of definite sign for all admissible displacement fields $\boldsymbol{\eta}$, or equivalently, for all possible ψ. This is certainly the case when $\Omega_o'(\Psi_o) < 0$. When $\Omega_o'(\Psi_o) > 0$, on the other hand, the two terms in (2.5) are of opposite sign, but nevertheless d^2E is negative definite (and the flow stable) for certain ranges of $\Omega_o'(\Psi_o)$. That is, for certain Ω_o', a routine eigenvalue problem furnishes the bound $\mathrm{d}^2E < -\lambda^2\int(\nabla\psi)^2\,\mathrm{d}V$.

We now consider Arnol'd's conserved-functional approach to stability. We shall see that this leads to precisely the same stability criterion as that given above. However, this second approach has the advantage of giving a simple explicit definition of stability which encompasses other more familiar forms. Following Arnol'd (1966*a*) we introduce the functional

$$A(\Psi) = \int_V \left[\tfrac{1}{2}(\nabla\Psi)^2 - \int_o^\Omega \Psi_o(\Omega)\,\mathrm{d}\Omega\right]\mathrm{d}V \tag{2.6}$$

where Ψ and Ω are the streamfunction and vorticity of some unsteady Euler flow. The function $\Psi_o(\Omega_o)$ is the inverse of $\Omega_o(\Psi_o)$, the vorticity distribution of some steady flow whose stability is under investigation, and $\Psi_o(\Omega)$ is a continuation of $\Psi_o(\Omega_o)$ outside the range of vorticity associated with the steady state. By virtue of the material conservation of vorticity and conservation of energy, $A(\Psi)$ is conserved by an unsteady flow. Now suppose that Ψ is close to Ψ_o, $\Psi = \Psi_o + \psi$. Then it is readily demonstrated that the first variation in A is zero. The second variation is

$$\delta^2A = \frac{1}{2}\int \left[(\nabla\psi)^2 - \left(\nabla^2\psi\right)^2/\Omega_o'(\Psi_o)\right]\mathrm{d}V. \tag{2.7}$$

But A is conserved by an Euler flow and so, in the linear approximation, $\delta^2 A$ is also conserved. If the flow is to be unstable (in some norm) then some suitable measure of the disturbance, $\|\psi\|^2$, must grow in time despite the conservation of $\delta^2 A$. In such cases the ratio $\delta^2 A/ \|\psi\|^2$ must tend to zero. Consequently, provided this ratio can be bound away from zero, the flow cannot become unstable in our chosen norm. (That is, linear stability is ensured if we can find bounds of the form $\delta^2 A < -\lambda^2 \|\psi\|^2$ or $\delta^2 A > \lambda^2 \|\psi\|^2$ for all possible ψ.) Flows for which $\delta^2 A$ is positive or negative definite are sometimes termed *formally stable* (Holm *et al.* 1985). This definition encompasses our conventional notion of stability, in which unstable normal modes grow exponentially (spectral stability).

We might (in passing) note that the δ-perturbation and d-perturbation are quite different. The latter is constrained to stay on an isovortical sheet while the former is unconstrained and encompasses all regions of function space surrounding an equilibrium point. One manifestation of this difference is the fact that $\mathrm{d}^1 E = 0$ while $\delta^1 E$ is, in general, non-zero. This difference is illustrated in figure 1(*b*).

Comparing (2.7) with (2.5) it is evident that $\delta^2 A = \mathrm{d}^2 E$, and so Arnol'd's two stability criteria are equivalent. Note that exponentially growing unstable modes must have a spatial structure which satisfies $\delta^2 A = 0$ (for fixed $\|\psi\|$). By implication, these unstable modes must satisfy $\mathrm{d}^2 E = 0$, which is the third constituent of Arnol'd's variational principle as stated in §1.

Now the equality $\delta^2 A = \mathrm{d}^2 E$ implies that if a flow is stable to the restricted class of isovortial perturbations then it is also linearly stable to non-isovortical perturbations. The reason is as follows. If we have a stable equilibrium on one isovortical sheet then in general there will be a neighbouring stable equilibrium on all adjacent sheets (Friedlander & Vishik 1990). Consequently, if a stable flow is perturbed onto an adjacent sheet it will find itself on a closed energy contour encircling the adjacent equilibrium. It then orbits the new equilibrium and so remains stable.

Now both of Arnol'd's methods are essentially variational techniques based on finding stationary values of the energy (or an energy-like functional). We might anticipate therefore that these different methods are fundamentally linked and we now show that this is indeed the case. In fact, as we shall see, and as suggested by Holm *et al.* (1985), the integral invariant (Casimir) which is added to E in the conserved-functional approach is simply a form of Lagrange multiplier which arises from the auxiliary constraint of material conservation of vorticity. Thus, although the 'δ-perturbation' appears to be quite general, in that we are not constrained to stay on an isovortical sheet, in both methods we are simply finding and classifying stationary values of E on an isovortical sheet. We may demonstrate this as follows.

Suppose we wish to find stationary values of E subject to vorticity being materially conserved during the variation. We could, on the one hand, follow Moffatt's method of using a virtual displacement field which explicitly builds in this constraint. Alternatively, we could follow Lagrange's method of the undetermined multiplier. In this approach we ignore the auxiliary constraints during the variation (in this case the material conservation of vorticity) but look for stationary values of a new functional comprising E plus an unknown multiplier times the auxiliary constraint. Provided the Lagrange multiplier is chosen correctly, the end result is the same. That is, we locate stationary values of E on an isovortical sheet. The only difference between the two methods is the manner in which the auxiliary constraint is handled. Now our auxiliary constraint is given by (2.2) in the form

$$\mathrm{d}^1 \Omega + \boldsymbol{\eta} \cdot \nabla \Omega_o = 0.$$

Thus stationary values of E on an isovortical sheet are given by

$$\mathrm{d}^1 E + \int \lambda(\boldsymbol{x}) \left[\mathrm{d}^1 \Omega + \boldsymbol{\eta} \cdot \nabla \Omega_o\right] \mathrm{d}V = 0$$

where $\lambda(\boldsymbol{x})$ is the Lagrange multiplier. We now choose our free parameter, $\lambda(\boldsymbol{x})$, so that the expression above is satisfied even when we lift the restriction imposed by the auxiliary constraint. That is, we choose λ so that our equation remains valid as we move from the restricted d-perturbation to the quite general δ-perturbation:

$$\int \left[\Psi_o \delta^1 \Omega + \lambda \left(\delta^1 \Omega + \boldsymbol{\eta} \cdot \nabla \Omega_o\right)\right] \mathrm{d}V = 0.$$

This must hold for all possible $\delta^1 \Omega$. Clearly the choice of $\lambda = -\Psi_o$ satisfies this requirement, since the final term in the integrand then disappears through the use of Gauss's theorem. It appears, therefore, that stationary values of E on an isovortical sheet are given by

$$\delta^1 E - \int \Psi_o \delta^1 \Omega \mathrm{d}V = 0.$$

We may rewrite this as

$$\delta^1 A = \delta^1 \left[E - \int \left[\int_o^\Omega \Psi_o(\Omega)\, \mathrm{d}\Omega\right] \mathrm{d}V\right] = 0.$$

Evidently, and as suggested by Holm *et al.* (1985) and Shepherd (1992), Arnol'd's two stability techniques represent different sides of the same coin. The integral invariant (or Casimir) which is added to E to form A is simply a form of a Lagrange multiplier. Nevertheless, the manner in which the methods are implemented is different and this is important. The advantage of the conserved-functional route is that it provides a simple clear definition of stability. On the other hand the variational (or energy) method is, in some sense, more fundamental in that it obviates the need to determine the appropriate form of Lagrange multiplier (Casimir) for each new type of flow. Moreover, as we shall see, the energy method also provides nonlinear stability criteria in precisely the same manner as the conserved-functional route.

2.2. *A generalization of Arnol'd's functional*

Now, with the aid of (2.1), Arnol'd's functional may be rewritten in an equivalent if unfamiliar form:

$$A = E - \int \left[\Omega \Psi_o(M) + C_o(M)\right] \mathrm{d}V. \tag{2.8}$$

Here C_o is the Bernoulli function of a steady flow whose stability is in question, and $M = \Omega$. It is readily confirmed that, in this new form, $\delta^1 A = 0$ and $\delta^2 A$ is given by (2.7). As we shall see, this apparently minor alteration is, in fact, of considerable significance. While (2.6) is particular to unforced planar incompressible flows, (2.8) is the appropriate Arnol'd functional for both unforced compressible flow and, more importantly, forced flows of the type governed by (1.1). It is necessary only to:

(i) reinterpret E as the net energy conserved by this flow;

(ii) replace M by the appropriate materially conserved quantity. In the case of flows governed by (1.1), $M = \Phi$. For unforced compressible flows, on the other hand, we have $M = \Omega/\rho$.

Subject to this generalization, the first variation in A always vanishes, while the second variation provides the appropriate linear stability criterion. For this reason, we shall refer to (2.8) as the generalized version of Arnol'd's functional. It provides a suitable starting point for establishing formal stability for a wide range of flows. Functionals of the form (2.8) have been used by Holm *et al.* to study both planar MHD flows (type 2) and compressible baratropic flow. However, as indicated above, (2.8) has a broader significance. Here our plan is to use (2.8), with M set equal to Φ, to investigate the formal stability of all forced flows governed by (1.1). We then show that precisely the same information is furnished by the variational (or energy) approach, and we take this as proof of Proposition 1.

3. Several classes of forced two-dimensional flows

We now show that a wide range of flows conform to (1.1)–(1.3). We are interested in both planar and poloidal motions and we start with the former. There are three planar flows of interest here: natural convection; MHD flows in which $\boldsymbol{B}$ is trapped in the plane of the motion; and flows driven by electrostatic forces. In the case of natural convection the body force per unit mass is

$$\boldsymbol{F} = -\left(\rho' g/\bar{\rho}\right)\hat{\boldsymbol{e}}_y = -\left(\rho' g/\bar{\rho}\right)\nabla y$$

where ρ' is the departure of the local density from the mean, $\bar{\rho}$. Here we employ the Boussinesq approximation, in which the fluid is incompressible and $\rho' \ll \bar{\rho}$. We ignore molecular diffusion so that ρ' is 'frozen' into the fluid; that is, $\mathrm{D}\rho'/\mathrm{D}t = 0$. Under these conditions our flow satisfies (1.1)–(1.3), with $\Phi = \rho' g/\bar{\rho}$, $f = y$ and $C = p/\rho + \boldsymbol{u}^2/2 + f\Phi$.

Next, consider an incompressible fluid of infinite electrical conductivity in which a planar magnetic field $\boldsymbol{B}(x, y) = \left(B_x, B_y, 0\right)$ is trapped. Since $\boldsymbol{B}$ is solenoidal we may introduce a flux function for $\boldsymbol{B}$, defined via

$$\boldsymbol{B} = (\rho\mu)^{1/2}\,\nabla\times[\Phi\hat{\boldsymbol{e}}_z]\,.$$

Here μ is the permeability of the medium. If we assume that $\boldsymbol{B}\cdot\boldsymbol{n}$ is zero on S then we may take $\Phi = 0$ at the boundary. The Lorentz force per unit mass may now be written as

$$\boldsymbol{F} = \frac{1}{\mu\rho}\,(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\left(\nabla^2\Phi\right)\nabla\Phi,$$

while the flux function satisfies $\mathrm{D}\Phi/\mathrm{D}t = 0$. Once again we have a system which conforms to (1.1)–(1.3). This time $f = -\nabla^2\Phi$, $C = p/\rho + u^2 2$, and the total kinetic energy of the fluid, E, is composed of magnetic plus kinetic energy.

Finally, let us consider motion driven by the mutual interaction of bound electric charges. Suppose we have an incompressible fluid of negligible electrical conductivity which contains a distribution of electric charge which is bound to the molecules. One example is air which contains small charged dust particles. If the particles are very fine (submicron in size) then they will move with the fluid. The charge density, q, is materially conserved. Moreover, if the fluid velocity is not too great then the only body force is the electrostatic one,

$$\boldsymbol{F} = -\left(q/\rho\right)\nabla V, \quad \nabla^2 V = -q/\epsilon_o.$$

Here V is the electrostatic potential and ϵ_o is the permeativity of the medium. Let us now define f as $\left(\epsilon_o/\rho\right)^{1/2} V$ and Φ as $q/(\rho\epsilon_o)^{1/2}$. Then our system conforms to (1.1)–(1.3), where $\nabla^2 f = -\Phi$ and $C = p/\rho + u^2/2 + f\Phi$. These results are summarized in table 1.

Type of flow	Φ	f	C	E
(i) Natural convection	$\Phi = \rho' g/\bar{\rho}$	$f = y$	$p/\rho + \boldsymbol{u}^2/2 + f\Phi$	$\int \left[f\Phi + \boldsymbol{u}^2/2 \right] \mathrm{d}V$
(ii) MHD flow	$\nabla \times [\Phi \hat{\boldsymbol{e}}_z] = \dfrac{\boldsymbol{B}}{(\rho\mu)^{1/2}}$	$f = -\nabla^2 \Phi$	$p/\rho + \boldsymbol{u}^2/2$	$\int \frac{1}{2} \left[f\Phi + \boldsymbol{u}^2 \right] \mathrm{d}V$
(iii) Electro-static flow	$\Phi = q/(\rho\epsilon_o)^{1/2}$	$\nabla^2 f = -\Phi$	$p/\rho + \boldsymbol{u}^2/2 + f\Phi$	$\int \frac{1}{2} \left[f\Phi + \boldsymbol{u}^2 \right] \mathrm{d}V$

TABLE 1. Forced, planar flows.

We now turn our attention to the poloidal flows (iv)–(vii) of §1. Consider first axisymmetric swirling flow. If Γ is the angular momentum $u_\theta r$, then the azimuthal component of the Euler equation tells us that Γ is materially conserved. Also, by virtue of the swirl, the poloidal component of motion experiences the inertial force associated with the centripetal acceleration. Provided we choose the initial condition to be such that Γ is zero on S, then this flow satisfies (1.1)–(1.3), where $\Phi = \Gamma^2$, $f = \left(2r^2\right)^{-1}$, and $C = p/\rho + \boldsymbol{u}^2/2 + f\Phi$.

Consider next an incompressible axisymmetric flow evolving under the influence of the mutual gravitational attraction of its component parts. Let the density of the fluid vary with position. As in example (i), the departure of the local density from the mean, $\bar{\rho}$, is assumed to be frozen into the fluid and much less than the mean, $\rho' \ll \bar{\rho}$. The gravitational potential, V, is determined by ρ according to $\nabla^2 V = 4\pi G\rho$, $V = \bar{V} + V'$. If we let $\Phi = \rho' G, f = \bar{V}/\bar{\rho}G$ and $C = p/\rho + \boldsymbol{u}^2/2 + \rho V/\bar{\rho}$, then this flow satisfies (1.1)–(1.3), where f is determined by $\nabla^2 f = 4\pi$.

Next, consider the poloidal flow of an incompressible perfectly conducting fluid which contains an azimuthal magnetic field, B_θ. The magnetic field is frozen into the fluid and evolves so that B_θ/r is materially conserved. Also, the Lorentz force per unit mass, $\boldsymbol{F}$, may be written in the form

$$(\rho\mu)\boldsymbol{F} = -\left(B_\theta/r\right)\nabla\left(B_\theta\, r\right) = -\nabla\left(B_\theta^2\right) + \left(r^2/2\right)\nabla\left(B_\theta/r\right)^2 .$$

If we now take $\Phi = B_\theta^2/\left(\rho\mu r^2\right)$, $f = r^2/2$ and $C = p/\rho + \boldsymbol{u}^2/2 + 2f\Phi$ then, once again, our system conforms to (1.1)–(1.3).

Finally, we consider a poloidal magnetic field which is locked into a perfectly conducting fluid and satisfies $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ on S. By analogy with the planar case we introduce the flux function defined by

$$\boldsymbol{B} = (\rho\mu)^{1/2}\,\nabla \times \left[\left(\Phi/r\right)\hat{\boldsymbol{e}}_\theta\right].$$

It is readily demonstrated that Φ is materially conserved, $\mathrm{D}\Phi/\mathrm{D}t = 0$, and that the Lorentz force per unit mass is

$$\boldsymbol{F} = -\nabla\cdot\left[(\nabla\Phi)/r^2\right]\nabla\Phi.$$

Once again, our flow conforms to (1.1)–(1.3), with $f = -\nabla\cdot\left[(\nabla\Phi)/r^2\right]$ and $C = p/\rho + \boldsymbol{u}^2/2$. These results are tabulated in table 2.

Type of flow	Φ	f	C	E
(iv) Swirling flow	$\Phi = \Gamma^2$	$f = \left(2r^2\right)^{-1}$	$p/\rho + \boldsymbol{u}^2/2 + f\Phi$	$\int \left[f\Phi + \boldsymbol{u}^2/2\right] \mathrm{d}V$
(v) Self-gravitating flow	$\Phi = \rho' G$	$\nabla^2 f = 4\pi$	$p/\rho + \boldsymbol{u}^2/2 + \rho V/\bar{\rho}$	$\int \left[f\Phi + \boldsymbol{u}^2/2\right] \mathrm{d}V$
(vi) MHD flow with B, θ-field	$\Phi = B_\theta^2/\left(\rho\mu r^2\right)$	$f = r^2/2$	$p/\rho + \boldsymbol{u}^2/2 + 2f\Phi$	$\int \left[f\Phi + \boldsymbol{u}^2/2\right] \mathrm{d}V$
(vii) MHD flow with poloidal field	$\nabla \times \left[\left(\Phi/r\right)\hat{\boldsymbol{e}}_\theta\right] = \dfrac{\boldsymbol{B}}{(\rho\mu)^{1/2}}$	$f = -\nabla \cdot \left[(\nabla\Phi)/r^2\right]$	$p/\rho + \boldsymbol{u}^2/2$	$\int \frac{1}{2}\left[f\Phi + \boldsymbol{u}^2\right] \mathrm{d}V$

TABLE 2. Forced, poloidal flows.

4. General properties of forced two-dimensional flow

As a prelude to our stability analysis we examine the more general properties of flows governed by

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\Omega} - \nabla C + f\nabla\Phi, \quad f = f(\boldsymbol{x}, \Phi). \tag{4.1}$$

We shall establish: (i) the energy properties of such flows; (ii) the steady-state solutions; (iii) the integral invariants; and (iv) a finite-amplitude conservation theorem for perturbations about a steady state. These are all essential prerequisites to our discussion of linear (and nonlinear) stability. We start with energy.

4.1. *Conservation of energy*

We may show that (4.1) describes a conservative system as follows. f is not an explicit function of time, and so the Lagrangian is a function only of $\boldsymbol{u}, \boldsymbol{x}$ and Φ. But Φ simply labels and locates the individual fluid particles. It follows that the Lagrangian is a function only of the generalized coordinates and velocities, and conservation of energy then follows. Now suppose we take the dot-product of $\boldsymbol{u}$ with (4.1). Then we obtain

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{u}^2}{2}\right) + f\frac{\partial \Phi}{\partial t} = -\nabla \cdot (C\boldsymbol{u}). \tag{4.2}$$

If, but only if, f is a prescribed function of $\boldsymbol{x}$ then (4.2) gives

$$E = \int \left[f\Phi + \boldsymbol{u}^2/2\right] \mathrm{d}V = \text{constant} \tag{4.3}$$

which is in accordance with tables 1 and 2. Flows for which f does not depend on time are an important subclass of (4.1) and deserve some special attention. This includes natural convection ($f = y$), swirling flow $\left(f = r^{-2}/2\right)$, self-gravitating flows $\left(\nabla^2 f = 4\pi\right)$, and MHD flows with an azimuthal field $\left(f = r^2/2\right)$. The distinguishing feature of these motions is that the force acting on *any one material element* is a conservative one, despite the fact that $\nabla \times \boldsymbol{F}$ is, in general, non-zero. That is, to within the gradient of a scalar, we may write $\boldsymbol{F}$ as

$$\boldsymbol{F} = -\Phi\nabla f\,(\boldsymbol{x}) \tag{4.4}$$

where Φ is frozen into each element of the fluid. Consequently, we are free to refer to f as a potential and to $\int f\Phi \mathrm{d}V$ as the potential energy of the fluid. We may regard the flow as evolving under a pseudo-gravitational potential, f, and with Φ playing the role of a pseudo-density. We might anticipate, therefore, that a generic feature of such flows is a predisposition to some kind of stratification and, as we shall see, this is in fact the case.

To see how this stratification might come about, consider (i) an isolated region of heavy material, (ii) an isolated circular magnetic flux tube and (iii) a hoop of swirling fluid. A blob of (relatively) heavy fluid in a uniform gravitational field will, of course, fall, lowering its potential energy. Similarly, a ring of swirling liquid in an otherwise quiescent fluid will centrifuge itself radially outward, thus lowering its azimuthal kinetic energy (Davidson 1994), and an azimuthal magnetic flux tube will, if unimpaired, collapse radially inward, reducing its magnetic energy. In these last two cases $\Phi = \Gamma^2$ and $\Phi \sim B_\theta^2/r^2$. When viewed in terms of (pseudo-gravitational) potentials each of these events is the same. 'Heavy' fluid (large $\Gamma, B_\theta/r$ or ρ') will free-fall through its potential field, lowering its potential energy. That is, in each case the movement is such as to lower f. There is a natural tendency, then, for these flows to become stratified with the 'heaviest' material at the lowest potential, although true stratification can never be reached because there is no dissipative mechanism available for destroying the kinetic energy liberated by the fall in potential energy.

Now the pseudo-gravitational interpretation of flows in which f is independent of time suggests that each of these motions has a minimum potential energy state corresponding to a vertical or radial stratification of Φ. In fact, the truth of this is readily demonstrated using the Schwarz inequality. Let E_f represent the energy associated with the body force. Then,

$$E_f \geqslant \frac{\left[\displaystyle\int \Phi^{1/2}\mathrm{d}V\right]^2}{\displaystyle\int f^{-1}\mathrm{d}V}. \tag{4.5}$$

We shall find this pseudo-gravitational viewpoint particularly useful in our interpretation of certain stability results. Specifically, we shall see that, whenever f is a prescribed function of position, the corresponding flow fails to meet Arnol'd's stability criterion. We argue that this is indicative of a real instability the origin of which lies in those regions of the flow where Φ is 'unstably' stratified (e.g. heavy fluid flowing over lighter fluid). The point is that confined steady solutions of (1.1) are of the form $\Phi = \Phi(\Psi)$ and so there will always exist regions where ρ', Γ or B_θ/r is unstably stratified. We shall see that potential energy may be released by perturbing the flow at such points, and consequently such regions are 'locally unstable' to short-wavelength perturbations. (We use the phrase 'locally unstable' to indicate that the amplitude of a local wave packet grows with time. See §9.) We argue that these local instabilities are carried on the back of the mean flow and that this lies at the root of the global instability.

So far we have discussed only flows in which f is a prescribed function of position. When f is also a function of Φ, and consequently an implicit function of time, the force acting on *any one material element* is no longer conservative, despite the fact that the overall system is conservative (Landau & Lifshitz 1959). In such instances tables 1 and 2 give

$$E = \frac{1}{2}\int \left[f\Phi + \boldsymbol{u}^2\right] \mathrm{d}V. \tag{4.6}$$

4.2. *Steady solutions*

In discussing the structure of steady flows it is convenient to distinguish between planar and poloidal motions. We start with the former. We shall exclude degenerate (stratified) solutions of the form $\boldsymbol{u} = 0$ and introduce the streamfunction, Ψ, defined by $\boldsymbol{u} = \nabla \times [\Psi \hat{\boldsymbol{e}}_z]$. Then, in the steady state, (1.3) and (4.1) yield

$$\Phi = \Phi(\Psi); \quad C = C(\Psi), \tag{4.7}$$

$$\Omega = -\nabla^2\Psi = f\Phi'(\Psi) - C'(\Psi). \tag{4.8}$$

Solutions of this type are well known in the context of MHD flow. Perhaps the simplest solutions correspond to $\Phi = \alpha^2\Psi^2$ and $C = \text{const}$. The streamfunction then satisfies

$$\nabla^2\Psi + \alpha^2(2f)\Psi = 0, \quad \Psi = 0 \quad \text{on} \quad S. \tag{4.9}$$

Note that the choice of α is not arbitrary, but is fixed by the shape of the domain. We shall refer to these as *linear flows*, in line with their governing equation.

The equivalent results for the poloidal flows are as follows:

$$\boldsymbol{u} = \nabla \times \left[(\Psi/r)\,\hat{\boldsymbol{e}}_\theta\right], \tag{4.10}$$

$$\Phi = \Phi(\Psi); \quad C = C(\Psi), \tag{4.11}$$

$$\Omega/r = -\nabla\cdot\left[(\nabla\Psi)/r^2\right] = f\Phi'(\Psi) - C'(\Psi). \tag{4.12}$$

In the interests of economy we have used the same symbol for the planar and Stokes streamfunctions, and also for the planar and azimuthal vorticity components. The simplest solution of (4.12) is $C = \text{const.}$, $f = f(\boldsymbol{x})$ and $\Phi = \alpha^2\Psi^2$. The streamfunction is then determined by the eigenvalue problem

$$\nabla\cdot\left[(\nabla\Psi)/r^2\right] + \alpha^2(2f)\,\Psi = 0, \quad \Psi = 0 \quad \text{on} \quad S. \tag{4.13}$$

We are interested in the stability of solutions of (4.8) and (4.12).

4.3. *Integral invariants and generalized isovortical sheets*

Although vorticity is not materially conserved by (4.1), it is preserved in a global sense, as we shall now show. Once again it is convenient to differentiate between planar and poloidal motion and we start with the former. Let us introduce the solenoidal vector field $\boldsymbol{H}$ defined by

$$\boldsymbol{H} = \nabla \times [\Phi\hat{\boldsymbol{e}}_z]. \tag{4.14}$$

Then the curl of (4.1) gives,

$$\frac{\mathrm{D}\Omega}{\mathrm{D}t} = \boldsymbol{H}\cdot\nabla f. \tag{4.15}$$

We now integrate (4.15) over the area bounded by two adjacent Φ-lines. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \Omega \mathrm{d}V = \oint f\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{S} = 0. \tag{4.16}$$

Evidently, the total vorticity contained between two adjacent Φ-lines is conserved (a form of Kelvin's theorem). It follows that the flow possesses integral invariants of the form

$$I(\Omega, \Phi) = \int_{V_\Phi} [\Omega g(\Phi) + h(\Phi)] \, \mathrm{d}V \tag{4.17}$$

where V_Φ is any material volume bounded by the surface $\Phi =$ constant. As already noted, invariants like (4.17) are essential for constructing conserved functionals of the form (2.8), and hence to establishing formal stability. Note that a special case of (4.17) is $\int \Omega\Phi \mathrm{d}V$, which is the cross-helicity of $\boldsymbol{u}$ and $\boldsymbol{H}$.

The equivalent results for poloidal flows are

$$\boldsymbol{H} = \nabla \times \left[\left(\Phi/r \right) \hat{\boldsymbol{e}}_\theta \right], \tag{4.18}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\Omega}{r}\right) = \boldsymbol{H}\cdot\nabla f, \tag{4.19}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V_\Phi}\left[\frac{\Omega}{r}g(\Phi) + h(\Phi)\right]\mathrm{d}V = 0. \tag{4.20}$$

As noted earlier, Arnol'd's variational (or energy) principle rests on the idea of unsteady Euler flows following constant-energy contours on an isovortical sheet. We now use invariants (4.17) and (4.20) to extend this idea to forced flows. Our aim is to define the generalized isovortical sheet (iso-Casimir surface) appropriate to (4.1) as a prelude to our discussion of Arnol'd's variational principle in §5. In the interests of brevity we restrict the discussion to planar motion.

Let us introduce the composite vector field $\boldsymbol{v}(x, y) = \left(u_x, u_y, \Phi\right)$. Now consider the function space of all solenoidal fields $\boldsymbol{v}$ which satisfy $\boldsymbol{u}\cdot\boldsymbol{n} = \Phi = 0$ on S. Flows governed by (4.1) follow a trajectory (evolve) in this function space. We now subdivide this space into lower-dimensional subdomains. To avoid inventing additional terminology we shall refer to these as isovortical sheets. These sheets (lower-dimensional subspaces) are defined in the following manner. Two flows $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are termed isovortical if:

(i) we can generate Φ_2 from Φ_1 by smoothly advecting Φ_1 using a volume-preserving displacement field, $\boldsymbol{\eta}(x, y)$, which satisfies $\boldsymbol{\eta}\cdot\boldsymbol{n} = 0$ on S;

(ii) during the application of displacement field $\boldsymbol{\eta}$, we preserve $\int \Omega \mathrm{d}V$ for each material strip enclosed by two adjacent Φ-lines, Φ and $\Phi + \delta\Phi$.

We now define each isovortical sheet to be composed of a particular set of isovortical flows. An unsteady motion governed by (4.1) obeys both constraints listed above and so evolves on such a sheet. Moreover, solutions of (4.1) conserve-energy and so follow constant-energy trajectories on a particular sheet. The situation is essentially that shown in figure 1.

We would expect steady solutions of (1.1) to represent stationary points on such a sheet, $\mathrm{d}^1E = 0$. Moreover, we might anticipate that extremal points represent stable steady flows, while saddle points represent (potentially) unstable flows. We shall confirm, in §5, that this viewpoint is indeed valid.

4.4. *A finite-amplitude conservation theorem*

We conclude this section by considering finite-amplitude perturbations to steady solutions of (4.1). We shall show that (1.1) and (1.3) support a finite-amplitude disturbance theorem. That is, there exists a 'wave quantity' $b(\boldsymbol{x}, t)$ which may be small or finite, is quadratic in the amplitude of the perturbation (for small amplitudes), and which is materially conserved to within the flux of a vector $\boldsymbol{F}$ which satisfies $\boldsymbol{F}\cdot\boldsymbol{n} = 0$ on S. This is the counterpart of the McIntyre & Shepherd (1987) finite-amplitude disturbance theorem for unforced Euler flows and is a prerequisite for the nonlinear stability theorems of §10. In the interests of brevity we restrict the discussion to planar flows.

Let e be the energy density of the flow, $\boldsymbol{u}^2/2+f\Phi$ or $\boldsymbol{u}^2/2+f\Phi/2$, and define $a(\boldsymbol{x},t)$, the density of Arnol'd's functional (2.8), as

$$a = e - \Omega\Psi_o(\Phi) - C_o(\Phi). \tag{4.21}$$

Here $\Psi_o(\Phi_o)$ and $C_o(\Phi_o)$ are the streamfunction and Bernoulli function of some steady flow around which we shall perturb. Next, we introduce a related parameter $b(\boldsymbol{x},t)$ defined as

$$b = (a - a_o) - \nabla\cdot[\Psi_o(\Phi_o)\nabla\psi] - \nabla\cdot(\boldsymbol{g}/2) \tag{4.22}$$

where $\psi = \Psi - \Psi_o$, and $\boldsymbol{g}$ is zero for natural convection, $\boldsymbol{g} = \phi\nabla\Phi_o - \Phi_o\nabla\phi$ for MHD flow, or $f_o\nabla\tilde{f} - \tilde{f}\nabla f_o$ for electrostatic flow where $\phi = \Phi - \Phi_o, \tilde{f} = f - f_o$. From (4.2), (4.6) and (4.15) it may be shown that b obeys a conservation equation of the form

$$\frac{\mathrm{D}b}{\mathrm{D}t} = \nabla\cdot[\boldsymbol{F}], \quad \boldsymbol{F}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad S. \tag{4.23}$$

Substituting for $a(\boldsymbol{x},t)$ in (4.22) gives an explicit expression for b:

$$b(\boldsymbol{x},t) = \frac{1}{2}(\nabla\psi)^2 + \nabla^2\psi\,[\Psi_o(\Phi) - \Psi_o(\Phi_o)] + \frac{1}{2}\tilde{f}\phi \tag{4.24}$$

$$+\nabla^2\Psi_o\left[\Psi_o(\Phi) - \Psi_o(\Phi_o) - \Psi_o'(\Phi_o)\,\phi\right] - \left[C_o(\Phi) - C_o(\Phi_o) - C_o'(\Phi_o)\,\phi\right]. \tag{4.25}$$

Evidently, b is globally conserved and is of quadratic order for small-amplitude disturbances. We shall use the conservation of b to develop nonlinear stability theorems in §10. First, however, we look at linear stability and Proposition 1.

5. The Kelvin–Arnol'd energy principle for forced flows

As noted in §4, forced conservative flows governed by (1.1) follow constant-energy contours on an isovortical sheet. By definition, we move from one flow to another on a sheet by smoothly advecting Φ with a solenoidal displacement field, $\boldsymbol{\eta}$, subject to the constraint that $\int\Omega\mathrm{d}V$ is conserved for every material volume bounded by two adjacent Φ-lines, Φ and $\Phi+\delta\Phi$. This definition ensures that all integrals of the type $\int_{V_\Phi}[\Omega g(\Phi) + h(\Phi)]\,\mathrm{d}V$ (sometimes called signature functions) are conserved by a flow as it evolves on the sheet. Proposition 1 asserts that steady flows are stationary points on such a sheet, $\mathrm{d}^1E = 0$. (As in §2, we use d rather than δ to represent an isovortical perturbation.) Moreover, we claim that stable steady flows represent extremums in E. In this Section and the next we give an explicit proof that this is indeed the case and, en route, we provide a simple universal recipe for evaluating d^2E. Our plan is to first evaluate d^2E on an isovortical sheet and then use Proposition 1 to establish general stability criteria for steady solutions of (1.1). Subsequently, in §6, we take the conserved-functional route. We then show that this yields precisely the same stability criteria, and we take this as proof of Proposition 1. We start with planar flows.

Consider the following perturbation. Suppose that, for a short time τ, we advect both Ω and Φ by a 'virtual' velocity field $\hat{\boldsymbol{v}}(x,y)$ (satisfying $\nabla\cdot\hat{\boldsymbol{v}} = 0$ and $\hat{\boldsymbol{v}}\cdot\boldsymbol{n} = 0$ on S). Simultaneously, we rearrange the vorticity field, but in such a way that total vorticity enclosed by each Φ-line remains constant. Both of these perturbations conserve the net vorticity contained within two adjacent Φ-lines, and so this represents a *generalized isovortical perturbation*, as defined in §1.

We can realize this perturbation using the following evolution equations:

$$\frac{\hat{\mathrm{D}}\Phi}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \hat{\boldsymbol{v}}\cdot\nabla\right)\Phi = 0; \quad 0 < t < \tau, \tag{5.1}$$

$$\frac{\hat{\mathrm{D}}\Omega}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \hat{\boldsymbol{v}}\cdot\nabla\right)\Omega = \boldsymbol{H}\cdot\nabla\zeta; \quad 0 < t < \tau, \tag{5.2}$$

where ζ is an arbitrary function of position, but not a function of time. (Expressions of this form have been used before for swirling and stratified flows. See Davidson (1994) and Vladimirov (1987).) Expression (5.1) satisfies the first requirement, that Φ be materially advected, while (5.2) fulfils the second requirement that the total vorticity enclosed by each Φ-line is conserved during the displacement. The similarity between (5.1) and (5.2), on the one hand, and the governing equations (4.15) and (1.3), on the other, is not accidental. These equations mimic the action of a real velocity field in perturbing Ω and Φ, although E is not, in general, conserved by (5.1) and (5.2). Note that we may vary the manner in which Ω is redistributed within each Φ-line by changing $\zeta(x,y)$. It follows that (5.1) and (5.2) are the most general form of isovortical perturbation. The application of these virtual displacements to a steady solution of (1.1) allows us to migrate at will across an isovortical sheet, exploring adjacent flows.

We now introduce $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$, defined by

$$\boldsymbol{\eta} = \hat{\boldsymbol{v}}\tau; \quad \boldsymbol{\lambda} = \nabla\times[\zeta\hat{\boldsymbol{e}}_z]\,\tau.$$

As in §2, $\boldsymbol{\eta}$ is a virtual displacement field, which satisfies $\nabla\cdot\boldsymbol{\eta} = 0$ and $\boldsymbol{\eta}\cdot\boldsymbol{n} = 0$ on S. From (5.1) and (5.2), the first- and second-order changes in Φ and Ω may be written in terms of $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ as follows:

$$\mathrm{d}^1\Phi = -\boldsymbol{\eta}\cdot\nabla\Phi, \tag{5.3}$$

$$\mathrm{d}^1\Omega = -\boldsymbol{\eta}\cdot\nabla\Omega - \boldsymbol{\lambda}\cdot\nabla\Phi, \tag{5.4}$$

$$\mathrm{d}^2\Phi = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla\left(\mathrm{d}^1\Phi\right), \tag{5.5}$$

$$\mathrm{d}^2\Omega = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla\left(\mathrm{d}^1\Omega\right) - \tfrac{1}{2}\boldsymbol{\lambda}\cdot\nabla\left(\mathrm{d}^1\Phi\right), \tag{5.6}$$

$$\mathrm{d}^1\boldsymbol{u} = \hat{\psi}\nabla\Omega + (\zeta\tau)\nabla\Phi + \nabla\phi_1, \tag{5.7}$$

$$\mathrm{d}^2\boldsymbol{u} = \tfrac{1}{2}\hat{\psi}\nabla\left(\mathrm{d}^1\Omega\right) + \tfrac{1}{2}(\zeta\tau)\nabla\left(\mathrm{d}^1\Phi\right) + \nabla\phi_2, \tag{5.8}$$

where $\hat{\psi}$ is the streamfunction for $\boldsymbol{\eta}$ and ϕ_1 and ϕ_2 are chosen to ensure $\mathrm{d}^1\boldsymbol{u}$ and $\mathrm{d}^2\boldsymbol{u}$ are solenoidal. Now, for the flows listed in table 1, the first-order change in energy is

$$\mathrm{d}^1E = \int\left(\boldsymbol{u}\cdot\mathrm{d}^1\boldsymbol{u} + f\mathrm{d}^1\Phi\right)\mathrm{d}V. \tag{5.9}$$

If we assume that $\boldsymbol{u}$ represents a steady solution of (1.1), and substitute for $\mathrm{d}^1\boldsymbol{u}$ and $\mathrm{d}^1\Phi$, we find $\mathrm{d}^1E = 0$. This confirms the first part of Proposition 1, at least for planar flows.

The second and third parts of Proposition 1 concern the second variation in E. This is given by

$$\mathrm{d}^2E = \int\left[\tfrac{1}{2}\left(\mathrm{d}^1\boldsymbol{u}\right)^2 + \boldsymbol{u}_o\cdot\mathrm{d}^2\boldsymbol{u} + f_o\mathrm{d}^2\Phi + \tfrac{1}{2}\mathrm{d}^1\Phi\;\mathrm{d}^1f\right]\mathrm{d}V \tag{5.10}$$

which, on substituting for $\mathrm{d}^2\boldsymbol{u}$ and $\mathrm{d}^2\Phi$ becomes

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\psi)^2 + (\zeta\tau)\,\boldsymbol{u}_o\cdot\nabla\phi + \nabla^2\psi\,(\boldsymbol{u}_o\cdot\nabla\hat{\psi}) - f_o\boldsymbol{\eta}\cdot\nabla\phi + \phi\tilde{f}\right]\mathrm{d}V.$$

Here we have introduced the notation $\phi = \mathrm{d}^1\Phi_o$, $\psi = \mathrm{d}^1\Psi_o$ and $\tilde{f} = \mathrm{d}^1 f_o$. Note that ϕ and ψ are kinematically independent scalar fields, in that ϕ is a function only of $\boldsymbol{\eta}$, while ψ depends on both ζ and $\boldsymbol{\eta}$. Note also that ϕ and ψ are not arbitrary but are governed by (5.3) and (5.4). We now use the divergence theorem, in conjunction with (4.8), to rewrite d^2E as

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\psi)^2 + 2\left(\nabla^2\psi\right)\epsilon + \left[\Phi_o''(\Psi_o)f_o - C_o''(\Psi_o)\right]\epsilon^2 + \phi\tilde{f}\right]\mathrm{d}V$$

where $\epsilon = \Psi_o'(\Phi_o)\,\phi = -\boldsymbol{\eta}\cdot\nabla\Psi_o$. The functions $\Phi_o(\Psi_o)$ and $C_o(\Psi_o)$ describe some steady solution of (1.1) whose stability is under investigation. Finally, we introduce γ, defined as $\gamma = \epsilon - \psi$. (Note that γ and ϵ are kinematically independent.) This allows us to express d^2E in terms of ϵ and γ, rather than ψ and ϕ:

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 - (\nabla\epsilon)^2 + g^*\epsilon^2 + \left(\Phi_o'\tilde{f}\right)\epsilon\right]\mathrm{d}V. \tag{5.11}$$

Here g^* contains all the residual information on the base flow, and is given by

$$g^* = f_o\Phi_o''(\Psi_o) - C_o''(\Psi_o). \tag{5.12}$$

Proposition 1 states that stable flows correspond to d^2E being positive or negative definite, and that exponentially growing normal modes must have a spatial structure which satisfies $\mathrm{d}^2E = 0$. We shall prove this assertion (via the conserved-functional route) in the next section. Here we merely note that (5.11) provides a stability test for all planar solutions of (1.1).

We shall postpone our detailed examination of (5.11) until §8, where we establish the conditions under which d^2E is of definite sign for all possible ϵ and γ. We shall see that, at least for certain types of flow, this leads to a sufficient condition for stability. In the meantime, we might note that whenever f is a prescribed function of position, so that (as far as each material element is concerned) the body force is conservative, d^2E takes the simple form

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 - (\nabla\epsilon)^2 + g^*\epsilon^2\right]\mathrm{d}V.$$

In such cases, d^2E is always indefinite in sign. For example, for short-wavelength disturbances the final term in the integrand can be neglected and so d^2E may be positive or negative depending on whether $\|\gamma\|$ is larger or smaller than $\|\epsilon\|$. We shall argue, in §8, that this failure to meet our stability criterion is indicative of a real instability.

The application of Arnol'd's variational principle to forced poloidal flows is, to all intents and purposes, the same as for planar flows. In the interests of brevity we will not repeat the arguments here. We merely note that the perturbation equations (5.1) and (5.2) are replaced by

$$\frac{\hat{\mathrm{D}}\Phi}{\mathrm{D}t} = 0; \quad 0 < t < \tau,$$

$$\frac{\hat{\mathrm{D}}}{\mathrm{D}t}\left(\frac{\Omega}{r}\right) = \boldsymbol{H}\cdot\nabla\zeta; \quad 0 < t < \tau,$$

and that the corresponding changes in energy are $d^1E = 0$ and

$$d^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 r^{-2} - (\nabla\epsilon)^2 r^{-2} + g^*\epsilon^2 + \left(\Phi_o'\tilde{f}\right)\epsilon\right] dV.$$

Here γ, ϵ, g^*, Φ_o' and $\tilde{f}$ are defined as for planar flows. Note that, as with the planar flows, d^2E is of indefinite sign whenever f is a prescribed function of position. This suggests instability.

6. An explicit proof of Proposition 1 via the conserved-functional method

We now apply Arnol'd's conserved-functional method to solutions of (1.1). We shall see that the resulting stability criteria are identical to those predicted by Proposition 1, and we take this as proof of our Proposition. The proof relies on the existence of the integral invariants (4.17) and (4.20), and is inspired by the generalization of Arnol'd's functional (2.8). We start with a second proposition.

Proposition 2. *Suppose $\Psi_o(\Phi_o)$ and $C_o(\Phi_o)$ are the streamfunction and generalized Bernoulli function of some steady solution of (1.1) whose stability is under investigation. Then, for planar flows, an appropriate generalization of Arnol'd's functional is*

$$A(\Psi,\Phi) = E - \int \left[\Omega\Psi_o(\Phi) + C_o(\Phi)\right] dV. \tag{6.1}$$

For poloidal flows, on the other hand, the appropriate form is

$$A(\Psi,\Phi) = E - \int \left[\left(\Omega/r\right)\Psi_o(\Phi) + C_o(\Phi)\right] dV. \tag{6.2}$$

In either case A is conserved by an unsteady flow, its first variation vanishes, and the sign of δ^2A provides information on the stability of the motion.

At this point we might make four comments. First, the invariance of A follows directly from (4.17) and (4.20). Second, in line with the notation in §4, Ω and Ψ take on slightly different meanings in the planar and poloidal cases. Third, we shall prove Propositions 1 and 2 only for the seven cases outlined in §3. However, it seems reasonable to expect this to hold for all solutions of (1.1). Finally, functionals similar to (6.1) have been used before for certain specific flows, most notably by Holm *et al.* in the investigation of planar MHD flow. However, we contend that (6.1) and (6.2) have a more universal significance, at least for flows of type (1.1), as we now show.

For convenience, we shall consider planar and poloidal motions separately. We start with planar flows. As before, we define ϕ, ψ and $\tilde{f}$ via the expressions $f = f_o + \tilde{f}$, $\Phi = \Phi_o + \phi$ and $\Psi = \Psi_o + \psi$. (For cases where f is a prescribed function of position, $\tilde{f} = 0$.) Of course, ϕ and ψ represent the departure of an unsteady flow from equilibrium. In a linear stability analysis we assume $\psi \ll \Psi_o$ and $\phi \ll \Phi_o$ although, for the moment, we need not make this approximation.

Let A_o be the value of A at equilibrium. Then it is not difficult to show that, for the flows listed in table 1,

$$\begin{aligned} A - A_o = \int \Big[& \tfrac{1}{2}(\nabla\psi)^2 + \nabla^2\psi\left[\Psi_o(\Phi_o+\phi) - \Psi_o(\Phi_o)\right] \\ & + \nabla^2\Psi_o\left[\Psi_o(\Phi_o+\phi) - \Psi_o(\Phi_o) - \Psi_o'(\Phi_o)\phi\right] \\ & - \left[\,C_o(\Phi_o+\phi) - C_o(\Phi_o) - C_o'(\Phi_o)\phi\right] + \tfrac{1}{2}\tilde{f}\phi\Big] dV. \end{aligned} \tag{6.3}$$

This form of $A - A_o$ is suitable for investigating nonlinear stability, where the expressions in brackets are replaced by the remainder term in a truncated Taylor series (see §10). An alternative means of establishing the invariant (6.3) is to use the finite-amplitude conservation theorem of §4.4. In any event, it is evident from the form of $A - A_o$ that, when ϕ and ψ are small, all first-order terms in the integrand vanish, and so $\delta^1 A = 0$. The second variation in A, on the other hand, is readily shown to be

$$\delta^2 A\,(\epsilon, \gamma) = \frac{1}{2}\int \left[(\nabla\gamma)^2 - (\nabla\epsilon)^2 + g^* \epsilon^2 + \left(\Phi_o' \tilde{f}\right)\epsilon\right] \mathrm{d}V. \tag{6.4}$$

Here we have replaced ψ and ϕ by $\epsilon = \Psi_o'(\Phi_o)\,\phi$ and $\gamma = \epsilon - \psi$. The function g^* contains the residual information about the base flow and is defined by (5.12). Note that $\delta^2 A = \mathrm{d}^2 E$.

Now A is conserved by an unsteady flow and so, in the linear approximation, $\delta^2 A$ is also conserved. Suppose that $\|(\psi, \phi)\|^2$ is some suitable measure of the disturbance, say

$$\|(\psi, \phi)\|^2 = \int (\nabla\psi)^2\,\mathrm{d}V + \int (\nabla\phi)^2\,\mathrm{d}V. \tag{6.5}$$

Then an unstable normal mode requires $\|(\psi, \phi)\|^2 \to \infty$ as $t \to \infty$. Thus, spectral instability requires $\delta^2 A / \|\psi, \phi\|^2 \to 0$. (We assume that linear instability sets in as some exponentially growing mode, and we exclude those cases where initial algebraic growth gives way to exponential decay at large times.) We conclude, therefore, that if $\delta^2 A$ can be bound away from zero for all kinematically admissible forms of ψ and ϕ, then the flow must be stable. In short, a sufficient condition for stability is that $\delta^2 A$ is positive or negative definite. Moreover, if the flow is unstable, exponentially growing modes have a spatial structure which ensures $\delta^2 A = 0$. This confirms Proposition 2 for planar flows. More importantly, though, we have also proved Proposition 1. The point is that $\delta^2 A = \mathrm{d}^2 E$ and so when $\delta^2 A$ is positive or negative definite, so is $\mathrm{d}^2 E$. It follows that a maximum or a minimum of E on an isovortical sheet does indeed ensure stability, as predicted by the second part of Proposition 1. Moreover, exponentially growing normal modes have a spatial structure which satisfies $\delta^2 A = 0$ and so they must also satisfy $\mathrm{d}^2 E = 0$. This confirms the third part of Proposition 1, at least for the planar flows listed in table 1. We suspect, but have not proved, that these results extend to all planar solutions of (1.1).

Let us now turn our attention to poloidal flows. If we linearize (6.2) about the steady flow (Ψ_o, Φ_o) then, as with the planar case, $\delta^1 A = 0$. The second variation in A is readily shown to be

$$\delta^2 A\,(\epsilon, \gamma) = \frac{1}{2}\int \left[(\nabla\gamma)^2 r^{-2} - (\nabla\epsilon)^2 r^{-2} + g^* \epsilon^2 + \left(\Phi_o' \tilde{f}\right)\epsilon\right] \mathrm{d}V \tag{6.6}$$

where, as before, $\epsilon = \Psi_o'(\Phi_o)\,\phi, \gamma = \epsilon - \psi$, and g^* is given by (5.12). Now A is an invariant and $\delta^1 A = 0$. Consequently, $\delta^2 A$ is conserved by any unsteady motion. Following the argument given for planar flows, stability then corresponds to $\delta^2 A$ being of definite sign, which confirms Proposition 2 for poloidal flows. More importantly, though, $\delta^2 A = \mathrm{d}^2 E$ and so stability corresponds to extremums in E on an isovortical sheet. This confirms Proposition 1 for the poloidal flows of table 2.

We conclude, therefore, that Propositions 1 and 2 do indeed hold for the seven classes of flow given in tables 1 and 2. Of course we have not yet determined which, if any, of the flows (1)–(vii) are stable. We have noted only that the cases where f is a prescribed function of position are potentially unstable. We leave a detailed investigation of the sign of d^2E until §8, where we examine stability on a case by case basis.

7. A third stability criterion

It is natural to enquire whether an energy principle of the type described in Proposition 1 is, in some simple way, related to Hamilton's principle of least action. We show here that a simple connection does indeed exist. We also describe a simple relaxation scheme for finding steady solutions of (1.1).

7.1. *A new stability criterion based on the Lagrangian*

PROPOSITION 3. *Systems described by (1.1) and (1.3) are stable whenever the Lagrangian is a maximum with respect to a virtual displacement of the particle trajectories.*

This might best be described as a principle of maximum (rather than least) action. It should be emphasized that the virtual displacement used here is different to that employed in either of the earlier stability theorems. Rather, it is of the type normally associated with Hamilton's principle, in which the system trajectory is perturbed in configuration space in such a way that the time of flight for each particle is preserved. Note that this proposition is, in effect, a direct extension of the stability criteria for static systems used, for example, in the magnetostatics of ideal fluids.

In order to prove Proposition 3 it is convenient to consider first a simpler system with a finite number of degrees of freedom. Suppose we have N interacting particles, each of fixed mass m_i and each with a particular (conserved) value of Φ, say Φ_i. Suppose moreover that these particles interact and evolve according to

$$m_i \frac{d\boldsymbol{u}_i}{dt} = -\Phi_i \nabla_i f_i(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots \boldsymbol{r}_N) \tag{7.1}$$

where $\boldsymbol{r}_i$ are the position vectors of the particles and the subscript on ∇ implies differentiation with respect to $\boldsymbol{r}_i$. Perhaps the simplest examples of such a system are: (i) a set of (slowly moving) charged masses subject to mutual electrostatic forces; and (ii) a self-gravitating set of masses.

Now f_i does not depend explicitly on time. Since f_i is uniquely determined by the instantaneous particle configuration the system is conservative with an energy, E, given by $E = T + V$, where T is the total kinetic energy and V is the potential energy associated with the force, $-\Phi\nabla f$. The Lagrangian for the system is $L = T - V$. Now Hamilton's principle tells us that, out of all the possible paths the system could travel from its initial position at time t_1 to its final position at t_2, it will actually follow the path for which the action integral

$$I = \int_{t_1}^{t_2} L\,dt$$

is stationary. Crucially, this is also true for the Lagrangian of each *individual* mass m_i. Now suppose that we perturb each particle one at a time, and that the true and perturbed trajectories are cyclic in the sense that all particle paths are closed and the particle velocities return to their original value on completing their cycles. Then provided the recirculation time, τ, is the same for the true and perturbed paths the action integral for each particle will be stationary. This is also true if we perturb the particle paths simultaneously, each with their separate travel times τ (Davidson 1994).

We must now translate these ideas into the language of fluid mechanics. Each fluid particle governed by (1.1) obeys an equation of the form of (7.1), except that we must add a pressure force to the right-hand side. However this pressure term is a 'force of constraint' whose function is to maintain conservation of volume. Provided that this constraint is observed during a virtual displacement the pressure force will do no net work (Davidson 1994). As such, it may be ignored in a Lagrangian formulation. If $\mathscr{L}$ is the Lagrangian density then, for each fluid particle moving around a closed streamline, we have

$$\mathrm{d}^1 \oint \mathscr{L} dt = 0$$

provided, of course, that the recirculation time is unchanged by the perturbation. Now consider a streamtube bounded by two adjacent streamlines, Ψ and $\Psi + \delta\Psi$. Then

$$\mathrm{d}t = d\ell / \,|\boldsymbol{u}| = \mathrm{d}V / \,|\delta\Psi\,|$$

where ℓ is the distance along the Ψ-line and $\mathrm{d}V$ is the volume of the streamtube associated with $\mathrm{d}\ell$. (For poloidal flows we must divide the right-hand side of this expression by a factor 2π.) Our integral with respect to time can now be replaced by a volume integral over the streamtube:

$$\mathrm{d}^1 \oint_{\Psi} \mathscr{L} \mathrm{d}V = 0.$$

Adding all such contributions from individual streamtubes we find

$$\mathrm{d}^1 \int_V \mathscr{L} \mathrm{d}V = \mathrm{d}^1 L = 0.$$

In summary, then, Hamilton's principle tells us that the (global) Lagrangian for the fluid is stationary provided: (i) the recirculation time for each streamline is preserved by the perturbation; (ii) the perturbation conserves volume; and (iii) Φ is materially conserved by each fluid particle during the perturbation.

We now demonstrate that stable solutions correspond to maxima in L. The first step is to find a perturbation which satisfies the three conditions above. Surprisingly, it turns out that such a perturbation is readily constructed.

Suppose that we perturb Ψ and Φ by the same virtual displacement field, $\boldsymbol{\eta}$. (As usual, we insist that $\nabla \cdot \boldsymbol{\eta} = 0$.) Then each fluid particle finds itself in a new trajectory, as defined by the new Ψ-lines, while retaining its original value of Φ. The second and third conditions above are therefore satisfied. Moreover, such an approach guarantees that the recirculation time, $\tau(\Psi)$, for each streamline is the same for the true and perturbed paths. This follows from

$$\tau(\Psi) = -\mathrm{d}V_{\Psi} / \mathrm{d}\Psi \tag{7.2}$$

where V_{Ψ} is the volume enclosed by a streamline. Conservation of V_{Ψ}, which is a consequence of $\nabla \cdot \boldsymbol{\eta} = 0$, automatically ensures conservation of τ. Our perturbation therefore satisfies all three requirements. We may now evaluate $\delta^1 L$ directly. As in §5, we imagine that the virtual displacement is achieved by advecting Ψ_o and Φ_o by an imaginary velocity field, $\boldsymbol{v}$, for a short period of time. This leads to

$$\mathrm{d}^1\Psi = \psi = -\boldsymbol{\eta} \cdot \nabla \Psi_o, \quad \mathrm{d}^2\Psi = -\tfrac{1}{2}\boldsymbol{\eta} \cdot \nabla \left(\mathrm{d}^1\Psi\right),$$

$$\mathrm{d}^1\Phi = \phi = -\boldsymbol{\eta} \cdot \nabla \Phi_o, \quad \mathrm{d}^2\Phi = -\tfrac{1}{2}\boldsymbol{\eta} \cdot \nabla \left(\mathrm{d}^1\Phi\right).$$

Note that, unlike the perturbation used in §5, ψ and ϕ are not independent, but are related by $\phi = \Phi_o'\psi$. For planar flows of the type listed in table 1, the first variation in L is

$$\mathrm{d}^1 L = \int [\nabla\Psi_o \cdot \nabla\psi - f_o\phi]\, \mathrm{d}V.$$

The divergence theorem, in conjunction with (4.8), then gives $\mathrm{d}^1 L = 0$. Thus, as required by Hamilton's principle, L is stationary. (The equivalent result for poloidal flows is readily established.)

Now consider the second variation in L. For the planar flows listed in table 1 this is given by

$$\mathrm{d}^2 L = \frac{1}{2}\int \left[(\nabla\psi)^2 - g^*\psi^2 - \left(\Phi_o'\tilde{f}\right)\psi\right] \mathrm{d}V.$$

The equivalent result for poloidal flows is

$$\mathrm{d}^2 L = \frac{1}{2}\int \left[r^{-2}(\nabla\psi)^2 - g^*\psi^2 - \left(\Psi_o'\tilde{f}\right)\psi\right] \mathrm{d}V.$$

Let us now change notation and write $\epsilon = -\boldsymbol{\eta}\cdot\nabla\Psi_o$ in line with §5. Then ϵ replaces ψ in our expressions for $\mathrm{d}^2 L$. If these expressions are now compared with (5.11) and (5.13), which give the change in E under an isovortical perturbation, then we find

$$\mathrm{d}^2 E = \frac{1}{2}\int (\nabla\gamma)^2\, \mathrm{d}V - \mathrm{d}^2 L \ (x, y \text{ motion}), \tag{7.3a}$$

$$\mathrm{d}^2 E = \frac{1}{2}\int (\nabla\gamma)^2 r^{-2} \mathrm{d}V - \mathrm{d}^2 L \ (r, z \text{ motion}). \tag{7.3b}$$

In either case $\mathrm{d}^2 E$ is positive definite whenever $\mathrm{d}^2 L$ is negative definite. It follows immediately that stability is ensured whenever L is a maximum, and this proves Proposition 3. We have, in effect, a new stability criterion. Remarkably, this holds not only for flows of type (i)–(vii) but is also true for three-dimensional Euler flows and, as we shall see in §8, it is true for flows driven by more than one body force. For example, natural convection subject to a magnetic field is stable whenever the Lagrangian is a maximum.

7.2. *Relaxation schemes for finding stable steady flows*

One of the advantages of Arnol'd's energy principle is that it leads quite naturally to relaxation schemes for finding stable steady solutions of (1.1), as we now demonstrate. The scheme described here is related to, but distinct from, that introduced by Shepherd (1992). Our starting point is the perturbation equations (5.1) and (5.2), or their equivalent for poloidal flow. Let us begin with the planar case.

Now we know that the evolution equations (5.1) and (5.2) cause some initial flow (Φ, Ω) to migrate across an isovortical sheet. Up until now we have considered small migrations, in which (5.1) is applied for a short time τ. Suppose we now consider large-scale migrations which occur for finite periods of time, according to

$$\frac{\hat{\mathrm{D}}\Phi}{\mathrm{D}t} = 0, \quad \frac{\hat{\mathrm{D}}\Omega}{\mathrm{D}t} = \boldsymbol{H}\cdot\nabla f, \tag{7.4}$$

$$\hat{\boldsymbol{v}} = \boldsymbol{u} + \alpha\frac{\partial\boldsymbol{u}}{\partial t}; \quad \alpha = \text{constant}. \tag{7.5}$$

We may think of $\hat{\boldsymbol{v}}$ as a continuously evolving form of Arnol'd's virtual displacement field. This particular system of equations is reminiscent of those proposed by Vallis, Carnevale & Young (1989) for finding stable steady solutions of the unforced Euler equations. They have some interesting properties. Specifically, while remaining on an isovortical sheet, they cause E to increase (for $\alpha < 0$) or decrease (for $\alpha > 0$) monotonically until such time that the flow reaches a steady state (assuming that a steady state exists on the sheet). Moreover, once a steady state is achieved, (7.4)–(7.5) revert to the forced system (1.1)–(1.3).

To demonstrate that E continually increases or decreases we first uncurl (7.4) to give

$$\frac{\partial \boldsymbol{u}}{\partial t} = \hat{\boldsymbol{v}} \times \boldsymbol{\Omega} - \nabla C + f\nabla \Phi. \tag{7.6}$$

The product of (7.6) with $\hat{\boldsymbol{v}}$ yields,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\alpha \int \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 \mathrm{d}V. \tag{7.7}$$

Evidently, (7.4)–(7.5) cause an arbitrary initial condition to migrate across an isovortical sheet, hunting out a maximum or minimum in energy. There are three likely outcomes of this procedure: $E \to 0$, $E \to \infty$, or an extremum in E is reached. In the latter case the relaxation equations provide a stable steady solution of (1.1). (A fourth possibility is that E remains finite but that there are no steady solutions on the sheet. This generally leads to infinite straining of the fluid.)

In order to guarantee non-trivial solutions we need to ensure that a steady solution lies on the sheet and find an upper or lower bound for E on each isovortical sheet. However, such bounds are readily obtained. For flows where f is a prescribed function of position (4.5) furnishes a lower bound, corresponding to a stratification of Φ. For planar MHD flow, on the other hand, we may use the calculus of variations to place a lower bound on E. That is,

$$E > E_f = \frac{1}{2}\int (\nabla\Phi)^2\,\mathrm{d}V \geqslant \lambda_o \frac{1}{2}\int \Phi^2 \mathrm{d}V \tag{7.8}$$

where λ_o is the least eigenvalue of

$$\nabla^2\Phi + \lambda\Phi = 0, \quad \Phi = 0 \text{ on } S\ .$$

The integral on the right is conserved on a sheet, and so (7.8) provides an appropriate lower bound for flows of type (ii). The equivalent results for poloidal MHD flow (case vii) are obtained by replacing Ω by Ω/r in (7.4) and using the bound

$$E > E_f = \frac{1}{2}\int \left[(\nabla\Phi)^2 r^{-2}\right] \mathrm{d}V \geqslant \lambda_o \frac{1}{2}\int \Phi^2 \mathrm{d}V \tag{7.9}$$

where λ_o is the minimum eigenvalue of

$$\nabla \cdot \left[\frac{1}{r^2}\nabla\Phi\right] + \lambda\Phi = 0, \quad \Phi = 0 \text{ on } S.$$

8. Stability criteria

We have not, so far, examined the conditions under which E is a maximum or a minimum, and so we have produced no specific stability criteria. We shall do this here. Of course, not all of the criteria that emerge are new. Many of these flows have been investigated before using the energy-Casimir method. However we do develop new criteria in §8.2. We start with the simple cases, where our aim is to show that our general expressions for d^2E give results compatible with known stability criteria.

8.1. *Familiar stability criteria*

We start with planar flows. We have, from §5,

$$d^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 - (\nabla\epsilon)^2 + g^*\epsilon^2 + \left(\Phi_o'\tilde{f}\right)\epsilon\right] dV \tag{8.1}$$

where $\tilde{f}$ takes the form: $\tilde{f} = 0$ (natural convection); $\tilde{f} = -\nabla^2\left[\Phi_o'\epsilon\right]$ (MHD flow); $\nabla^2\tilde{f} = -\Phi_o'\epsilon$ (electrostatic flow). Note that γ and ϵ are kinematically independent in the sense that $\epsilon = \epsilon(\boldsymbol{\eta})$ and $\gamma = \gamma(\boldsymbol{\eta}, \zeta)$. For natural convection, d^2E simplifies to

$$d^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 - (\nabla\epsilon)^2 + g^*\epsilon^2\right] dV. \tag{8.2}$$

As noted earlier, this is of indefinite sign for short-wavelength disturbances. All such flows are therefore potentially unstable. The most that we can deduce from (8.2) is that exponentially growing normal modes must have a spatial structure which satisfies $d^2E = 0$. It is not difficult to show that d^2E is also indefinite in sign for electrostatic flows.

Consider now MHD flow (case ii). Here d^2E takes the form

$$d^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 - (\nabla\epsilon)^2 + g^*\epsilon^2 + \left(\nabla\left(\Phi_o'\epsilon\right)\right)^2\right] dV.$$

This may be rearranged to give

$$d^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 + \left[1 - \left(\Psi_o'\right)^2\right](\nabla\phi)^2 + \hat{g}\phi^2\right] dV \tag{8.3}$$

where

$$\hat{g} = \Psi_o'\nabla^2\Psi_o' + \Psi_o''\nabla^2\Psi_o - C_o''(\Phi_o). \tag{8.4}$$

Stability criteria are readily extracted from (8.3). Evidently, a necessary condition for d^2E to be single signed is $|\Psi_o'| < 1$: that is, $|\boldsymbol{u}|$ is less than the Alfvén velocity. Indeed both Holm *et al.* (1985) and Vladimirov *et al.* (1996) note that a sufficient condition for stability is $|\Psi_o'| < 1$, and $\hat{g} > 0$. In fact, we can go further than this and identify stable steady flows in which $\hat{g} < 0$. Consider the eigenvalue problem

$$\nabla^2\phi + \lambda h\phi = 0, \quad \phi = 0 \quad \text{on} \quad S. \tag{8.5}$$

If we equate h to $|\hat{g}|/\left[1 - \left(\Psi_o'\right)^2_{max}\right]$, then it is not difficult to show that the last two contributions to the integral in (8.3) may be bounded from below by a positive number whenever the least eigenvalue λ_o of (8.5) is greater than unity. Thus, stability of type (ii) flows is ensured whenever

$$|\Psi_o'| < 1 \quad \text{and} \quad \hat{g} > 0 \tag{8.6a}$$

or

$$|\Psi_o'| < 1, \quad \hat{g} < 0, \quad \text{and} \quad \lambda_o \geqslant 1. \tag{8.6b}$$

In either case stable steady flows represent a minimum energy state. (A specific example of a flow whose stability rests on condition (8.6*b*) is given by $\Psi_o' = \alpha$, and $C_o'' = \lambda$. Here α and λ are constants and λ is the least eigenvalue of $\nabla^2\Psi_o +$

$\left(1-\alpha^2\right)^{-1}\lambda\Psi_o = 0$.) In §10 we shall show that these linear criteria may be extended to finite-amplitude perturbations.

We now consider the poloidal flows (iv)–(vii). Here d^2E takes the form

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 r^{-2} - (\nabla\epsilon)^2 r^{-2} + g^*\epsilon^2 + \Phi_o'\tilde{f}\epsilon\right]\mathrm{d}V \tag{8.7}$$

where $\tilde{f} = 0$ for all types of flow other than (vii). In cases where $\tilde{f} = 0$, d^2E simplifies to

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 r^{-2} - (\nabla\epsilon)^2 r^{-2} + g^*\epsilon^2\right]\mathrm{d}V \tag{8.8}$$

which is of indefinite sign. As with natural convection, the most that we can deduce from (8.8) is that exponentially growing normal modes must have a spatial structure that satisfies $\mathrm{d}^2E = 0$. We consider such modes in more detail in §9.

Finally we turn to poloidal MHD flows of type (vii). Here (8.7) becomes

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 r^{-2} - (\nabla\epsilon)^2 r^{-2} + g^*\epsilon^2 + \left(\nabla\left(\Phi_o'\epsilon\right)\right)^2 r^{-2}\right]\mathrm{d}V. \tag{8.9}$$

This may be rearranged to give

$$\mathrm{d}^2E = \frac{1}{2}\int \left[(\nabla\gamma)^2 r^{-2} + \left[1-\left(\Psi_o'\right)^2\right](\nabla\phi)^2 r^{-2} + \hat{g}\phi^2\right]\mathrm{d}V \tag{8.10}$$

where, this time, $\hat{g}$ is defined as

$$\hat{g} = \Psi_o'\nabla\cdot\left[r^{-2}\nabla\Psi_o'\right] + \Psi_o''\nabla\cdot\left[r^{-2}\nabla\Psi_o\right] - C_o''\left(\Phi_o\right). \tag{8.11}$$

If we compare this with the planar case then it is evident that the stability of type (vii) flows is guaranteed whenever conditions (8.6*a*) and (8.6*b*) above are satisfied. The only difference is that we must set $h = |\hat{g}|\,r^2/\left[1-\left(\Psi_o'\right)^2_{max}\right]$ in (8.5). Stability criteria for poloidal flows are discussed in Vladimirov, Moffatt & Illin (1997).

8.2. *New stability criteria*

As indicated in §2, the energy and conserved-functional approaches to stability are closely related. However, the advantage of the energy formulation lies in its generality. For example, if we accept that Proposition 1 is true for all solutions of (1.1), and not just for the seven cases discussed here, then it is also true for flows driven by composite body forces of the form $f_1\nabla\Phi_1 + f_2\nabla\Phi_2$. (This may be proven by the energy-Casimir method in the manner of §6, although we will not pause to do so here.) It is then a trivial matter to extend the method to more complex systems. Consider, for instance, planar natural convection in the presence of a planar magnetic field. (The equivalent axisymmetric problem has been considered by Friedlander & Vishik 1990.) We know that natural convection alone is (probably) unstable, while a magnetic field can be stabilizing. We might anticipate, therefore, that a sufficiently strong magnetic field will stabilize the flow. We have, in effect, a combination of type (i) and type (ii) flows. Let Φ_1 and Φ_2 represent the density and magnetic flux functions respectively, and f_1 and f_2 be their corresponding potentials. Steady flows satisfy a generalization of (4.8):

$$\Omega = f_1\Phi_1' + f_2\Phi_2' - C'(\Psi). \tag{8.12}$$

Now Φ_1 and Φ_2 are constant along streamlines in the steady state. Consequently, in the unperturbed state isolines of Φ_1 and Φ_2 coincide. Moreover, Φ_1 and Φ_2 are both displaced with the fluid particles during the initial perturbation, and since Φ_1

and Φ_2 are subsequently frozen into the fluid we have $\Phi_1 = \Phi_1(\Phi_2)$ at all times. This is a sufficient condition for the existence of integral invariants of the type (4.17). Consequently, we can determine the isovortical perturbations using a simple extension of the evolution equations (5.1) and (5.2):

$$\frac{\hat{\mathrm{D}}\Phi_1}{\mathrm{D}t} = \frac{\hat{\mathrm{D}}\Phi_2}{\mathrm{D}t} = 0, \quad \frac{\hat{\mathrm{D}}\Omega}{\mathrm{D}t} = \boldsymbol{H}_2 \cdot \nabla\zeta; \quad 0 < t < \tau. \tag{8.13}$$

Note that we have used $\boldsymbol{H}_2$, the scaled magnetic field, on the right-hand side of the vorticity equation (8.13). However, we could equally use $\boldsymbol{H}_1$, since contours of Φ_1, and Φ_2 are both frozen into the fluid during the perturbation. Following the procedure outlined in §5 it is readily confirmed that $\mathrm{d}^1 E = 0$, while $\mathrm{d}^2 E$ is given by

$$\mathrm{d}^2 E = \frac{1}{2}\int \left[(\nabla\gamma)^2 + \left((\Phi_2')^2 - 1\right)(\nabla\epsilon)^2 + \hat{g}^*\epsilon^2\right]\mathrm{d}V. \tag{8.14}$$

Here γ and ϵ are defined in the same way as before ($\epsilon = -\boldsymbol{\eta}\cdot\nabla\Psi_o, \gamma = \epsilon - \psi$) and $\hat{g}^*$ is given by

$$\hat{g}^* = \Phi_1'' f_1 + \Phi_2'' f_2 - C''(\Psi) - \Phi_2'\nabla^2\Phi_2'. \tag{8.15}$$

Evidently, stability corresponds to $\Phi_2' > 1$ (the flow is sub-Alfvénic) and $\hat{g}^* > 0$. No doubt we could obtain the same criterion by other means. However, the key point is that this stability criterion has been obtained quite simply and automatically and without the need to hunt for the appropriate Casimir.

We conclude by noting that the results of §7 also extend to flows driven by a composite body force. That is, the flow is stable whenever the Lagrangian L is a maximum under a variation of the particle trajectories in configuration space. We may establish this using the same procedure as outlined in §7. In the interests of brevity we omit the details here. We merely note that, following the logic of §7, we may show:

$$\mathrm{d}^2 E = \frac{1}{2}\int (\nabla\gamma)^2\,\mathrm{d}V - \mathrm{d}^2 L \quad \text{(planar flow)},$$

$$\mathrm{d}^2 E = \frac{1}{2}\int (\nabla\gamma)^2 r^{-2}\mathrm{d}V - \mathrm{d}^2 L \quad \text{(poloidal flow)}.$$

Thus, whenever $\mathrm{d}^2 L$ is negative definite, $\mathrm{d}^2 E$ is positive definite, and from Proposition 1 stability is ensured (provided we accept Proposition 1 in its most general form). It is remarkable that stability may be determined for such a broad class of flows by examining only L. It is tempting to speculate that there is some general principle underlying this result.

9. Stability of flows in which f is a prescribed function of position

We now consider those cases where f is a prescribed function of position. So far, we have noted the following properties of these motions.

(i) They all fail to meet Arnol'd's energy criterion, and so are potentially unstable.

(ii) As far as each fluid particle is concerned, the body force is conservative, with a potential f, and the total energy is composed of kinetic plus potential energy. By analogy with natural convection, we may think of f as a pseudo-gravitational field and Φ as a (materially conserved) pseudo-density. (See §4.1.)

(iii) The potential energy may be minimized by placing the 'heaviest' fluid (large Φ) at regions of lowest potential, and this results in a radial or vertical stratification of Φ. (See equation (4.5).)

(iv) Steady solutions (other than degenerate stratified ones) are of the form $\Phi = \Phi(\Psi)$ and so in simply connected domains there are always regions where Φ (the pseudo-density) increases with increasing potential.

Intuitively, we would expect all flows of this type to be unstable, with the instability rooted in the region of adverse stratification (where Φ increases with increasing f). We now argue that this is indeed the case. We do not provide a formal proof, but rather provide a simple physical explanation for why an instability is likely.

The argument is as follows. As we shall see, short-wavelength disturbances are convected by the mean flow, in the sense that the phase and group velocities of a wave packet are much less than the characteristic speed of the base flow. Moreover, such disturbances grow exponentially in regions of 'adverse' stratification and oscillate in a neutral manner in regions of 'stable' stratification. The fact that the streamlines are closed, so that individual fluid elements are continually swept through both regions, suggests (but does not guarantee) that any disturbance will ultimately grow, violating the requirements of formal stability.

We now justify this picture. Our starting point is to note that E may be divided into kinetic and potential energy. The contribution to d^2E from the potential energy is

$$\mathrm{d}^2E_f = \int f_o \mathrm{d}^2\Phi \mathrm{d}V = -\frac{1}{2}\int (\boldsymbol{\eta}\cdot\nabla f_o)(\boldsymbol{\eta}\cdot\nabla\Phi)\,\mathrm{d}V. \tag{9.1}$$

Now suppose the displacement field, $\boldsymbol{\eta}$, is a local rotation in the (x, y)- or (r, z)-plane, applied over a very small area centred on (x_o, y_o) or (r_o, z_o). Then, for natural convection, swirling flow, and flow driven by an azimuthal magnetic field, we obtain

$$\mathrm{d}^2E_f = \frac{1}{4}\Pi_i \int \boldsymbol{\eta}^2 \mathrm{d}V, \tag{9.2}$$

$$\Pi_1 = -\left[\frac{g}{\bar{\rho}}\frac{\partial \rho'}{\partial y}\right]_o \quad \text{(square of the Väisälä–Brunt frequency)}, \tag{9.3}$$

$$\Pi_4 = +\left[\frac{1}{r^3}\frac{\partial \Gamma^2}{\partial r}\right]_o \quad \text{(Rayleigh's discriminant)}, \tag{9.4}$$

$$\Pi_6 = -(\rho\mu)^{-1}\left[r\frac{\partial}{\partial r}\left(\frac{B_\theta}{r}\right)^2\right]_o. \tag{9.5}$$

Each of these expressions for Π represents the square of the frequency of oscillation of neutral modes in a quiescent stably stratified fluid. Adverse stratification corresponds to negative Π. Evidently, potential energy may be released by perturbing the flow in regions of negative Π: that is, where ρ' increases with y, Γ^2 decreases with r, or $|B_\theta/r|$ increases with r. We might anticipate that short-wavelength disturbances grow in such regions and we shall see that this is indeed the case.

Perhaps this behaviour is not surprising. It is well known that parallel horizontal flow which is subject to a vertical stratification of density, or a swirling pipe flow $(0, u_\theta, (r), u_z(r))$, is stable to short-wavelength disturbances if and only if the appropriate Π is positive. That is, the horizontal (or axial) component of motion has no influence on the stability of the flow, at least for short wavelengths. Stability is controlled simply by the stratification of ρ or Γ. Now in our case a small wave packet of very short wavelength is unaware of the curvature of the streamlines provided it is not too close to a stagnation point. It sees only a (locally) parallel flow in which the streamlines are aligned with the surfaces of constant density or angular momentum. We would expect, therefore, that such a perturbation will grow exponentially whenever Π_i is (locally) negative. We now prove this. In the interests of brevity we restrict the discussion to planar flows.

Let $\Phi = \Phi_o + \phi$ and $\Psi = \Psi_o + \psi$. Then for planar flows, (4.15) and (1.3) yield

$$\frac{D\phi}{Dt} = \left(\frac{\partial}{\partial t} + \boldsymbol{u}_o \cdot \nabla\right)\phi = \Phi_o'(\Psi_o)\,\boldsymbol{u}_o \cdot \nabla\psi,$$

$$\frac{D}{Dt}\left(\nabla^2\psi\right) + \boldsymbol{u}_o \cdot \nabla\left(g^*\psi\right) = \left[\nabla\left(\phi - \Phi_o'\psi\right) \times \nabla f\right]_z .$$

It is convenient to replace ϕ by $\gamma = \Psi_o'\phi - \psi$, and to equate f_o to y in line with flows of type (i). In this case our equations simplify to

$$\frac{D\gamma}{Dt} = -\frac{\partial\psi}{\partial t}, \tag{9.6}$$

$$\frac{D}{Dt}\left(\nabla^2\psi\right) + \boldsymbol{u}_o \cdot \nabla\left(g^*\psi\right) = \frac{\partial}{\partial x}\left(\Phi_o'\gamma\right).$$

We now eliminate ψ to give

$$\frac{D}{Dt}\left[\left(\nabla^2 + g^*\right)\frac{D\gamma}{Dt}\right] = \frac{\partial}{\partial t}\left[g^*\frac{D\gamma}{Dt} - \frac{\partial}{\partial x}\left[\Phi_o'\gamma\right]\right]. \tag{9.7}$$

Now suppose we look for a short-wavelength solution of the form

$$\gamma = \gamma_o \exp\left[\mathrm{i}\left(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}\right)\right]. \tag{9.8}$$

Here $\boldsymbol{k}$ is real and γ_o is the (slowly varying) amplitude of the wave packet. Let ℓ be a typical geometric length scale for the base flow, say $\ell \sim (g^*)^{-1/2}$. Since we are interested in short-wavelength disturbances, we have $k\ell \gg 1$. Then, to leading order in $(k\ell)^{-1}$, (9.7) gives the local dispersion equation

$$\omega_r k = \pm\left[-\Phi_o' k_x \boldsymbol{k} \cdot \boldsymbol{u}\right]^{1/2}, \quad \omega_r = \omega - \boldsymbol{k} \cdot \boldsymbol{u}. \tag{9.9}$$

Here ω_r is the wave frequency measured in a frame moving with the mean flow. Evidently, the wave is 'locally unstable', in the sense that it grows exponentially, whenever $\Phi_o' k_x\,(\boldsymbol{k} \cdot \boldsymbol{u})$ is positive. This is not unexpected since

$$\Phi_o' k_x\,(\boldsymbol{k} \cdot \boldsymbol{u}) = -\Pi_1 k_x^2\left(\boldsymbol{k} \cdot \boldsymbol{u}/k_x u_x\right) \tag{9.10}$$

where Π_1 is the square of the Väisälä–Brunt frequency (as defined in (9.4)). Negative values of Π_1 correspond to an adverse (unstable) stratification of density, and to complex values of ω_r.

The equivalent results for swirling flows are

$$r\omega_r k = \left(\Phi_o' k_z \boldsymbol{k} \cdot \boldsymbol{u}\right)^{1/2} \tag{9.11}$$

and

$$\Phi_o' k_z\,(\boldsymbol{k} \cdot \boldsymbol{u}) = r^2 \Pi_4 k_z^2\left(\boldsymbol{k} \cdot \boldsymbol{u}/k_z u_z\right). \tag{9.12}$$

This time a local instability sets in whenever Π_4 is negative. Again, this is not unexpected since Π_4 is Rayleigh's discriminant, and regions of negative Π_4 correspond to an adverse stratification of angular momentum.

In either case, $\omega_r \ll \boldsymbol{k}\cdot\boldsymbol{u}$, so that $\omega_r/k \ll |\boldsymbol{u}|$. The phase and group velocities of any wave packet are therefore much less than the characteristic velocity of the base flow. We may think of a disturbance as being carried on the back of the mean flow, oscillating in a neutral way in regions of 'stable' stratification and growing exponentially in regions of 'unstable' stratification. Now the fact that the wave is 'frozen' into the fluid has kinematic implications. In fact, it is not difficult to show that the kinematic (eikonal) equations for such a wave require that $\boldsymbol{k}\cdot\boldsymbol{u}$ is conserved (to leading order in $(k\ell)^{-1}$) by the packet, so there is no substantial change in the (streamwise) wavelength as the wave packet is swept around. This is a direct consequence of the wave having negligible group velocity (see Lighthill 1978).

We are left, therefore, with the following picture. Suppose a small localized disturbance is initiated in, say, a region of stable stratification and with $\boldsymbol{k}$ parallel to $\boldsymbol{u}$. Then this short-wavelength disturbance will be swept around by the base flow, passing successively through regions of 'stable' and 'unstable' stratification. As the disturbance is swept around it stays localized since the group velocity (the rate of propagation of wave energy) is much less than the convective velocity. In some regions the disturbance oscillates without loss or gain in potential energy. In the other regions the disturbance grows, converting potential energy into kinetic energy. Although this does not constitute a proof of instability it does seem likely that the flow will progressively depart from its equilibrium configuration, and that the failure to meet Arnol'd's stability criterion is indicative of a true instability.

A model problem which captures the spirit of such an instability is

$$\ddot{x} + \omega^2 \sin(\epsilon\omega t)\ x = 0, \quad \epsilon \ll 1.$$

This represents an oscillator whose natural frequency varies periodically on a slow time scale from ω to $\mathrm{i}\omega$. This is reminiscent of a disturbance being swept around in one of our flows, with $(\epsilon\omega)^{-1}$ representing the turnover time of an eddy. It is readily confirmed (see Appendix A) that the amplitude of oscillation for x grows by $\mathrm{e}^{2.4/\epsilon}$ for each of the (long-time-scale) cycles, and so our model problem is unstable.

10. Nonlinear stability

We turn now to the topic of nonlinear stability. In the interests of brevity we restrict attention to planar MHD flow, although essentially the same arguments may be applied to poloidal MHD flow. We shall demonstrate that the linear criteria associated with (8.3) and (8.4) extend in a simple way to finite-amplitude disturbances, although we must restrict the analysis to initial disturbances which are isovortical. We also demonstrate that the Kelvin–Arnol'd method is readily extended to finite-amplitude perturbations. We start, though, with the conserved-functional approach.

Recall that (6.3) is valid for finite values of ψ and ϕ. In effect, this follows from our finite-amplitude conservation theorem established in §4.4. We now replace the bracketed expressions in (6.3) by the remainder terms of the appropriate Taylor series. This yields an integral invariant, ΔA, of arbitrary magnitude

$$A - A_o = \Delta A = \int \left[\tfrac{1}{2}(\nabla\psi)^2 + \Psi_o'\phi\nabla^2\psi + \tfrac{1}{2}\Psi_o''\phi^2\nabla^2\Psi_o - \tfrac{1}{2}C_o''\phi^2 + \tfrac{1}{2}(\nabla\phi)^2\right] \mathrm{d}V. \quad (10.1)$$

The derivatives Ψ_o', Ψ_o'' and C_o'' are evaluated at certain (unknown) points in the range $\Phi_o < \Phi < \Phi_o + \phi$. Next, we rearrange the terms in the integrand to give expressions reminiscent of (8.3) and (8.4):

$$\Delta A = \frac{1}{2}\int \left[(\nabla\gamma)^2 + \left[1 - \left(\Psi_o'\right)^2\right](\nabla\phi)^2 + \hat{g}\phi^2\right] \mathrm{d}V, \tag{10.2}$$

$$\hat{g} = \Psi_o'\nabla^2\Psi_o' + \Psi_o''\nabla^2\Psi_o - C_o''. \tag{10.3}$$

Here $\Psi_o' = \Psi_o'(\Phi_o + \phi_1)$, $\Psi_o'' = \Psi_o''(\Phi_o + \phi_2)$ and $C_o'' = C_o''(\Phi_o + \phi_3)$ where ϕ_1, ϕ_2 and ϕ_3 all lie in the range $0 < \phi_i < \phi$. As before, γ is defined as $\Psi_o'\phi - \psi$, which, in the present context, is more meaningfully written as $\Psi_o(\Phi) - \Psi$. The only difference between (10.3) and its linear counterpart lies in the location at which Ψ_o', Ψ_o'' and C_o'' are evaluated. We now introduce bounds on Ψ_o' and $\hat{g}$. Let us suppose that there exist constants g, G, d and D such that, for all possible ϕ_i in the range 0–ϕ,

$$0 < d \leqslant \left(\Psi_o'\right)^2 \leqslant D < 1, \tag{10.4}$$

$$0 < g \leqslant \hat{g} \leqslant G < \infty. \tag{10.5}$$

Conditions (10.4) and (10.5) are sufficient to ensure linear stability. We shall now demonstrate that they also guarantee nonlinear stability. The first step is to use (10.4) and (10.5) to bound ΔA from above and from below:

$$\int \left[(\nabla\gamma)^2 + (1-D)(\nabla\phi)^2 + g\phi^2\right] \mathrm{d}V \leqslant 2\Delta A \leqslant \int \left[(\nabla\gamma)^2 + (1-d)(\nabla\phi)^2 + G\phi^2\right] \mathrm{d}V.$$

Next we must choose a measure of the size of the disturbance. It is convenient to use the norm

$$\|(\gamma,\phi)\|^2 = \int \left[(\nabla\gamma)^2 + (1-D)(\nabla\phi)^2 + g\phi^2\right] \mathrm{d}V. \tag{10.6}$$

Since ΔA is conserved we have, for all t,

$$\|(\gamma,\phi)\|^2 \leqslant \int \left[(\nabla\gamma)_o^2 + (1-d)(\nabla\phi)_o^2 + G\phi_o^2\right] \mathrm{d}V \tag{10.7}$$

where the subscript o indicates terms evaluated at $t = 0$. This is the key result. It implies that the magnitude of the disturbance, as measured by $\|(\gamma,\phi)\|$, is limited by the initial size of the perturbation (as measured in a slightly different way). To establish nonlinear stability in a formal way it remains to relate the integral on the right of (10.7) to our norm at $t = 0$. It is readily confirmed that

$$\int \left[(\nabla\gamma)_o^2 + (1-d)(\nabla\phi)_o^2 + G\phi_o^2\right] \mathrm{d}V \leqslant k^2 \ \|(\gamma,\phi)\|_o^2$$

where $k^2 = G/g$ if $(1-D)G > (1-d)g$, or $k^2 = (1-d)/(1-D)$ if $(1-D)G < (1-d)g$. This allows us to rewrite (10.7) in the form

$$\|(\gamma,\phi)\|^2 \leqslant k^2 \ \|(\gamma,\phi)\|_o^2 \tag{10.8}$$

where k is a known constant. We conclude, therefore, that $\|(\gamma,\phi)\|^2$ is bounded above by (a constant times) its initial value. Moreover, the magnitude of the disturbance can be bounded no matter how large its initial value. This establishes nonlinear stability, not only in the classic sense of Lyapunov, but also in a stronger sense. (See McIntyre & Shepherd (1987) for a discussion of the different definitions of non-linear stability.) In summary, then, provided Φ_o' and $\hat{g}$ can be bounded in accordance with (10.4) and (10.5), the flow is stable to disturbances of arbitrary magnitude. We have, in effect, extended the linear stability criterion into the nonlinear regime, provided that the initial perturbations are isovortical. (Note, however, that while nonlinear stability implies linear stability, the converse is not in general true, since the derivatives in the expression for $\hat{g}$ are evaluated at different points in the two cases.) Although we have restricted the argument to cases where $\hat{g} > 0$, it is a simple matter to show that the linear stability criterion (8.6) extends to nonlinear perturbations even when $\hat{g}$ is negative. (See Appendix B.)

Similar but distinct arguments have been used by Holm *et al.* (1985) and by Vladimirov *et al.* (1996), although these authors arrive at slightly different conclusions. In the case of Vladimirov *et al.* a different norm is employed for the linear and nonlinear cases and consequently their nonlinear analysis leads to a stability criterion which is not readily related back to the linear one. (Vladimirov *et al.* also consider the more difficult case of non-isovortical initial perturbations.) Holm *et al.* also use a norm (slightly) different to (10.6), and as a result their criterion is different from, though very similar to, ours. However, as noted by Vladimirov *et al.*, the method of proof employed by Holm *et al.* contains a flaw, although it is probable that this technical difficulty could be remedied by reformulating the problem.

Note that we have relied on the conserved-functional technique for establishing nonlinear stability. However, we can obtain precisely the same results using the variational (or energy) method. Indeed, in his original paper Arnol'd (1966*b*) notes that a nonlinear criterion may be obtained for Euler flows by considering finite-amplitude disturbances on an isovortical sheet. It turns out that the same is true of MHD flows, as we now show. Consider the change in energy resulting from a finite-amplitude disturbance on a generalized isovortical sheet. This is given by

$$\Delta E = \int \left[\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}(\nabla\psi)^2 - \phi\nabla^2\Phi_o - \Psi_o\nabla^2\psi\right] \mathrm{d}V$$

where ϕ and ψ are determined by (5.1) and (5.2) in the form

$$\frac{\partial\Phi}{\partial t} = -\hat{\boldsymbol{v}}\cdot\nabla\Phi, \quad 0 < t < \tau,$$

$$\frac{\partial\Omega}{\partial t} = -\hat{\boldsymbol{v}}\cdot\nabla\Omega + \boldsymbol{H}\cdot\nabla\zeta, \quad 0 < t < \tau.$$

Unlike §5, however, we now consider τ to be of finite magnitude. Combining the first of these expressions with a first-order truncated Taylor series for $\Phi(t)$, we obtain

$$\phi = \Phi(\tau) - \Phi_o = -\boldsymbol{\eta}\cdot\nabla\left[\Phi\left(t^*\right)\right]; \quad 0 < t^* < \tau.$$

More generally, if $F(\Phi)$ is a smooth function of Φ we have

$$F(\Phi_o + \phi) - F(\Phi_o) = -\boldsymbol{\eta}\cdot\nabla\left[F\left(\Phi\left(t^*\right)\right)\right]; \quad 0 < t^* < \tau.$$

Consequently, if we now represent $F(\Phi_o + \phi)$ by a second-order truncated Taylor series, we find

$$F'(\Phi_o)\,\phi = -\nabla\cdot\left[F\left(\Phi\left(t^*\right)\right)\boldsymbol{\eta}\right] - \tfrac{1}{2}F''\left(\Phi\left(t^{**}\right)\right)\phi^2; \quad 0 < t^{**} < \tau.$$

We now choose $F = C_o$ and this allows us, with the help of (4.8), to rewrite the third term in the expression for ΔE, $-\phi\nabla^2\Phi_o$, as

$$-\tfrac{1}{2}\phi^2\left(\Psi_o''\Omega_o + C_o''\right) - \nabla\cdot\left[C_o\boldsymbol{\eta}\right] + \Omega_o\left[\Psi_o(\Phi_o + \phi) - \Psi_o(\Phi_o)\right].$$

We shall return to this expression shortly. In the meantime we introduce γ, defined as before in the form

$$\gamma = \Psi_o'\left(\Phi\left(\hat{t}\right)\right)\phi - \psi = \Psi_o(\Phi) - \Psi.$$

It follows directly from this expression that

$$(\nabla\psi)^2 = (\nabla\gamma)^2 + \phi^2\left[\Psi_o'\nabla^2\Psi_o'\right] - \left(\Psi_o'\right)^2(\nabla\phi)^2 + 2\omega\left[\Psi_o(\Phi_o+\phi) - \Psi_o(\Phi_o)\right]$$
$$+\nabla\cdot\left[2\phi\Psi_o'\nabla\psi - \phi^2\Psi_o'\nabla\Psi_o'\right].$$

Substituting for $(\nabla\psi)^2$ and $\phi\nabla^2\Phi_o$ in our equation for ΔE gives

$$\Delta E \;=\; \frac{1}{2}\int\left[(\nabla\gamma)^2 + \left(1-\left(\Psi_o'\right)^2\right)(\nabla\phi)^2 + \hat{g}\phi^2\right]\mathrm{d}V$$

where $\hat{g}$ is defined by (10.3). As in the expression for ΔA, the exact points at which Ψ_o', Ψ_o'' and C_o'' are evaluated are unknown. Note the similarity to (10.2). Now conditions (10.4) and (10.5), which ensure nonlinear stability via the conserved-functional argument, are equivalent to the statement that $\Delta E > 0$ for all conceivable isovortical perturbations, whatever their magnitude. This is exactly what we would expect from Arnol'd's energy (i.e. variational) argument. That is, if the equilibrium flow represents an absolute minimum in energy on a generalized isovortical sheet then the flow must be nonlinearly stable in an energy norm. We have, in effect, shown how to generalize the Kelvin–Arnol'd method to nonlinear stability.

11. Conclusions

We have identified the steady solutions of (1.1)–(1.3). This represents a broad class of forced two-dimensional flows. The stability of these motions has been investigated using a variant of the Kelvin–Arnol'd energy principle. In line with previous investigators we find that the planar and poloidal MHD equations support a variety of (linearly) stable steady flows. The situation is quite different when f is a prescribed function of position. In this case the corresponding flows fail to satisfy the (generalized) Kelvin–Arnol'd energy criterion. We argue that this is indicative of a real instability of the Rayleigh–Taylor type. We have also shown that a new test for linear stability may be formulated in terms of the Lagrangian. Specifically, a flow is stable if its Lagrangian is a maximum for the steady state. All of these results may be extended to flows driven by more than one body force. In this context we have derived a new stability criterion for natural convection in a magnetic field. Finally, we have investigated the nonlinear stability of planar MHD flows. Our primary finding is that a simple modification of the well-known linear-stability criterion also provides a sufficient condition for nonlinear stability, provided the initial perturbations are isovortical. We also show how the Kelvin–Arnol'd method may be extended to obtain nonlinear stability criteria.

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Appendix A. Model problem of §9

Consider the equation

$$\ddot{x} + \omega^2\sin(\epsilon\omega t)\,x = 0, \quad \epsilon \ll 1.$$

This has differing solutions depending on whether the sin function is positive or negative. That is, for $-\pi < \epsilon\omega t < 0$,

$$x = [-\sin\epsilon\omega t]^{-1/4}\left[Ae^{\phi} + Be^{-\phi}\right], \quad \phi = \frac{1}{\epsilon}\int_o^{\epsilon\omega t}[-\sin\eta]^{1/2}\,d\eta$$

and for $0 < \epsilon\omega t < \pi$,

$$x = [+\sin\epsilon\omega t]^{-1/4}\left[a\cos\theta + b\sin\theta\right], \quad \theta = \frac{1}{\epsilon}\int_o^{\epsilon\omega t}[\sin\eta]^{1/2}\,d\eta.$$

These solutions are valid to leading order in ϵ but break down near $t = 0$. A different form of solution holds in the vicinity of $t = 0$ and matching this to the (outer) solution above leads to the well-known connection formulae

$$2\sqrt{2}A = (a+b), \quad \sqrt{2}B = a - b.$$

Now if we take as our initial conditions $a = -b$ at $\epsilon\omega t = \pi/2$, and follow the solution over one period, matching first at $\epsilon\omega t = \pi$ and then at $\epsilon\omega t = 2\pi$, we find

$$x\left(5\pi/2\right) = x\left(\pi/2\right)e^{2\theta_o}, \quad \epsilon\theta_o = \int_o^{\pi/2}[\sin\eta]^{1/2}\,d\eta = 1.20.$$

Evidently, the amplitude of the disturbance increases by $e^{2.4/\epsilon}$ per (long-time-scale) cycle. Of course this increase in amplitude simply represents the growth during the 'unstable' parts of the cycle.

Appendix B. Nonlinear stability of MHD flow

We note that a nonlinear stability criterion for class (ii) flows may also be established for those cases where $\hat{g}$ is negative. Suppose that $\hat{g} < 0$ and that there exist constants g, G, d and D such that

$$0 < d \leqslant \left(\Psi_o'\right)^2 \leqslant D < 1, \quad 0 < g \leqslant |\hat{g}| \leqslant G < \infty.$$

Moreover, suppose that the minimum eigenvalue, λ_o, of

$$\nabla^2\phi + \lambda\left[G/(1-D)\right]\phi = 0, \quad \phi = 0 \quad \text{on} \quad S$$

is greater than unity. As noted in §8, these conditions are sufficient to ensure linear stability. We now show that they also guarantee nonlinear stability provided the initial perturbations are isovortical. We start by placing bounds on ΔA according to

$$\int\left[(\nabla\gamma)^2 + (1-D)(\nabla\phi)^2 - G\phi^2\right]dV \leqslant 2\Delta A \leqslant \int\left[(\nabla\gamma)^2 + (1-d)(\nabla\phi)^2 - g\phi^2\right]dV.$$

Moreover, we have a second bound.

$$\int\left[(\nabla\phi)^2 - \left(G/(1-D)\right)\phi^2\right]dV \geqslant \left[(\lambda_o - 1)/\lambda_o\right]\int(\nabla\phi)^2\,dV > 0.$$

This enables us to replace the lower bound on ΔA to give

$$\int\left[(\nabla\gamma)^2 + \alpha^2(\nabla\phi)^2\right]dV < 2\Delta A \leqslant \int\left[(\nabla\gamma)^2 + (1-d)(\nabla\phi)^2 - g\phi^2\right]dV$$

where $\alpha^2 = (1-D)(\lambda_o - 1)\lambda_o^{-1}$. Finally we introduce the norm,

$$\|(\gamma,\phi)\|^2 = \int\left[(\nabla\gamma)^2 + \alpha^2(\nabla\phi)^2\right]dV.$$

Conservation of ΔA in conjunction with the expression above now provides us with an upper bound on our norm:

$$\|(\gamma, \phi)\|^2 \leqslant \int \left[(\nabla\gamma)_o^2 + (1-d)(\nabla\phi)_o^2 - g\phi_o^2\right] \mathrm{d}V.$$

Clearly, the magnitude of the disturbance, as measured by $\|(\gamma, \phi)\|^2$, is limited by the initial size of the perturbation, albeit measured in a different way.

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